

Solutions to complex smoothing equations

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Abstract We consider smoothing equations of the form

$$X \stackrel{\text{law}}{=} \sum_{j \geq 1} T_j X_j + C$$

where (C, T_1, T_2, \dots) is a given sequence of random variables and X_1, X_2, \dots are independent copies of X and independent of the sequence (C, T_1, T_2, \dots) . The focus is on complex smoothing equations, i.e., the case where the random variables X, C, T_1, T_2, \dots are complex-valued, but also more general multivariate smoothing equations are considered, in which the T_j are similarity matrices. Under mild assumptions on (C, T_1, T_2, \dots) , we describe the laws of all random variables X solving the above smoothing equation. These are the distributions of randomly shifted and stopped Lévy processes satisfying a certain invariance property called (U, α) -stability, which is related to operator (semi)stability. The results are applied to various examples from applied probability and statistical physics.

Keywords Branching process · characteristic function · infinite divisibility · Lévy process · multiplicative martingales · multivariate smoothing equation · similarity matrix

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1 Introduction and main results

1.1 Smoothing equations

In the paper at hand, we consider complex smoothing equations of the form

$$X \stackrel{\text{law}}{=} \sum_{j \geq 1} T_j X_j + C \quad (1.1)$$

where $\stackrel{\text{law}}{=}$ denotes equality in law, (C, T_1, T_2, \dots) is a given sequence of complex random variables and X_1, X_2, \dots are independent copies of the complex random variable X and independent of (C, T_1, T_2, \dots) . The law of X is called a solution¹ to (1.1). Under suitable assumptions on (C, T_1, T_2, \dots) , we provide a complete description of the set of all solutions to (1.1).

This kind of problem has a long history going back at least to [35, 41, 50]. Recently, there has been progress leading to a complete description of the set of all solutions to (1.1) in the case where all the T_j , $j \in \mathbb{N}$ are real and C and X are random vectors [5, 43]. We also refer to [2] for an overview over the early results in the field.

The complex case can naturally be embedded into the more general multivariate case, in which C and X are d -dimensional and the T_j are random $d \times d$ matrices, $d \in \mathbb{N}$. In this setup, (1.1) has been solved under two different sets of assumptions, namely, in the homogeneous case (i.e., $C = 0$ a.s.) under assumptions that guarantee that all solutions are essentially scale mixtures of multivariate normal distributions [14], and in the case that C, T_1, T_2, \dots have positive entries only and attention is restricted to solutions X on $[0, \infty)^d$ [64]. Yet, these results either cannot be applied to the complex case [64] or cover only a very special situation [14]. We will solve (1.1) in a multivariate setup that comfortably covers the complex case and thereby address open questions in papers by Barral [10, Remark 4] (posed for $C = 0$ and T_1, T_2, \dots taking values in a Banach algebra); Chauvin et al. [29, Remark 4.5] and Madaule et al. [61, Section 1.2] (in the complex case).

1.2 Motivation

We believe that solving (1.1) in a setup as general as possible is of high theoretical value for probability theory. Smoothing equations appear naturally in various fields of probability and statistical physics. We mention here two classes of examples that will also motivate the choice of our setup.

1.2.1 Complex smoothing equations in models of Applied Probability

In various models of Applied Probability including b -ary search trees [29, 30, 37, 46], Pólya urn models [46, 53, 69], the conservative fragmentation model [47],

¹ In slight abuse of language, we will sometimes call a random variable X a solution if the distribution of X is a solution to (1.1).

and B-urns [28] phase transitions were shown for the limiting behavior of the distributions of quantities of interest. All these phase transitions follow very similar patterns suggesting that they are particular instances of one universal phenomenon.

Before we sketch the general phenomenon, we describe it in the context of b -ary search trees. The space requirement (i.e., the total number of nodes) in a b -ary search tree with n keys inserted under the random permutation model, centered by its mean and scaled by \sqrt{n} , is known [58] to be asymptotically normal when $b \leq 26$. On the other hand, it exhibits stable periodic fluctuations [30, 37] around its mean when $b > 26$.

The general scheme is as follows. For each of the models mentioned above, a characteristic equation can be formulated with several complex roots (for b -ary search trees, b appears as a parameter in the equation). The root with largest real part always is 1. The asymptotic behavior of the quantity of interest is determined by the root with second largest real part, $\xi + i\eta$, say. Roughly speaking, if $\xi \leq \frac{1}{2}$, the fluctuations around the expected size of the quantity are of order \sqrt{n} (times possibly a slowly varying correction term), where n denotes the “size” of the model in an appropriate sense, and the limiting distribution is normal. When $\xi > \frac{1}{2}$, the fluctuations around the mean are of order n^ξ and there is no convergence but a periodic limiting behavior the precise description of which involves solutions to complex smoothing equations. Our results reflect the phase transition between normal and periodic limiting behavior with stable fluctuations. Concrete examples are considered in detail in Section 2.

1.2.2 Kinetic models

A main objective in the kinetic theory of gas is to understand the distribution μ_t on \mathbb{R}^3 of particle velocities and its evolution in time, given by the equation

$$\frac{\partial}{\partial t} \mu_t + \mu_t = S(\mu_t). \quad (1.2)$$

Eq. (1.2) has to be understood in the weak sense, i.e.,

$$\frac{\partial}{\partial t} \int f d\mu_t + \int f d\mu_t = \int f dS(\mu_t)$$

for all bounded continuous functions f on \mathbb{R}^3 . Here S is a mapping on probability measures, called collisional-gain operator, that describes the change in μ_t due to the collision of two uncorrelated particles. Its form can be deduced from the equations of energy and momentum conservation (see [73] for a detailed survey on the topic).

In order to understand qualitative aspects of (1.2), simplifications of S have been considered, the most important one being the Kac caricature of the Boltzmann equation [48], where μ_t is restricted to the real line, and

$$S : \mu \mapsto \text{law of } (\sin(\Theta)V_1 + \cos(\Theta)V_2)$$

where V_1, V_2 are i.i.d. with law μ and assumed to be independent of the random arc length Θ which has the uniform distribution on $[0, 2\pi)$. In this case, only energy is conserved, but not momentum. Any steady state distribution μ_∞ , i.e., satisfying $\frac{\partial}{\partial t}\mu_\infty = 0$, is a solution of the smoothing equation $S(\mu_\infty) = \mu_\infty$ with real valued weights. Such equations were investigated in [43].

Recently, there has been a lot of interest in generalizations of the Kac caricature [11, 12, 22], also with applications in economic theory [27, 32, 63], to mention just a few.

Of particular interest for us is a generalization of the Kac model in \mathbb{R}^3 , considered by Bassetti and Matthes [14, Section 6.2]. Here, a random vector V the law of which is the steady state distribution, satisfies

$$V \stackrel{\text{law}}{=} LV_1 + RV_2, \quad (1.3)$$

where V, V_1, V_2 are i.i.d. and independent of the random pair (L, R) of similarity matrices² which is subject to the condition $\mathbb{E}[\|L\|^2 + \|R\|^2] = 1$ and density assumptions, which guarantee that there is a unique solution (up to scaling) to (1.3). It is shown that this solution is a mixture of multivariate Gaussian distributions. We solve (1.3) in much greater generality. In particular, dropping their density assumption, we show that larger classes of Gaussian solutions appear.

1.3 Setup and main results

We aim for solving (1.1) in a multivariate setup covering both, the complex equation and the multivariate equation as outlined in Section 1.2.2.

Fix $d \in \mathbb{N}$, denote by $|x| = (x_1^2 + \dots + x_d^2)^{1/2}$ for $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ the Euclidean norm and by $\|a\| := \sup\{|ax| : |x| = 1\}$ for a real $d \times d$ -matrix a the associated matrix norm. Assume that we are given a sequence $(C, T_1, T_2, \dots) =: (C, T)$ where C is a d -dimensional random vector and $T = (T_j)_{j \in \mathbb{N}}$ is a sequence of random $d \times d$ similarity matrices, i.e., for each $j \in \mathbb{N}$, $T_j = \|T_j\| O_j$ where $O_j \in \mathbb{O}(d)$, with $\mathbb{O}(d)$ denoting the group of orthogonal $d \times d$ matrices. Notice that $\|T_j\|$ and O_j are random elements of $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{O}(d)$, respectively, which in general depend on each other. Further, we point out that no assumptions on the dependence structure of the sequence (C, T) are imposed.

This setup includes the case where X, C, T_1, T_2, \dots are complex random variables, henceforth referred to as the *complex case*, since \mathbb{C} can be identified with \mathbb{R}^2 and multiplication by a complex number $re^{i\theta}$, $r \geq 0$, $\theta \in [0, 2\pi)$ corresponds to multiplication in \mathbb{R}^2 from the left by the similarity matrix

$$\begin{pmatrix} r \cos \theta & -r \sin \theta \\ r \sin \theta & r \cos \theta \end{pmatrix}.$$

² By a $d \times d$ *similarity matrix* we mean a $d \times d$ matrix that can be written as a scale multiple of an orthogonal matrix.

We proceed by introducing the assumptions and some basic concepts needed for the statement of our main result.

1.3.1 Assumptions

Throughout the paper, we assume that the number of non-zero T_j is a.s. finite, i.e., $N := \#\{j \in \mathbb{N} : \|T_j\| > 0\} < \infty$ a.s. We suppose without loss of generality that $T_1, \dots, T_N \neq 0$ and $T_j = 0$ for $j > N$. For the formulation of further assumptions, we define, for $s \geq 0$,

$$m(s) := \mathbb{E} \left[\sum_{j=1}^N \|T_j\|^s \right]. \quad (1.4)$$

Notice that $m(0) = \mathbb{E}[N]$. Throughout the paper the following assumptions will be in force.

$$\mathbb{E}[N] > 1. \quad (\text{A1})$$

$$m(\alpha) = 1 \text{ for some } \alpha > 0. \quad (\text{A2})$$

(Notice that we do not exclude the case that $m(s) = \infty$ for all $s \neq \alpha$.) Then $W_1 := \sum_{j=1}^N \|T_j\|^\alpha$ is a nonnegative random variable with unit mean. In our main results, we assume that

$$m'(\alpha) := \mathbb{E} \left[\sum_{j=1}^N \|T_j\|^\alpha \log \|T_j\| \right] \in (-\infty, 0) \text{ and } \mathbb{E}[W_1 \log^+ W_1] < \infty. \quad (\text{A3})$$

Notice that $m'(\alpha)$ as defined in (A3) is indeed the derivative of $s \mapsto m(s)$ at α if the latter exists. α can be thought of as a generalized index of stability. Indeed, strictly α -stable random variables solve the particular instance of (1.1) with $C = 0$ and $T_j = N^{-1/\alpha}$, $N \geq 2$ fixed.

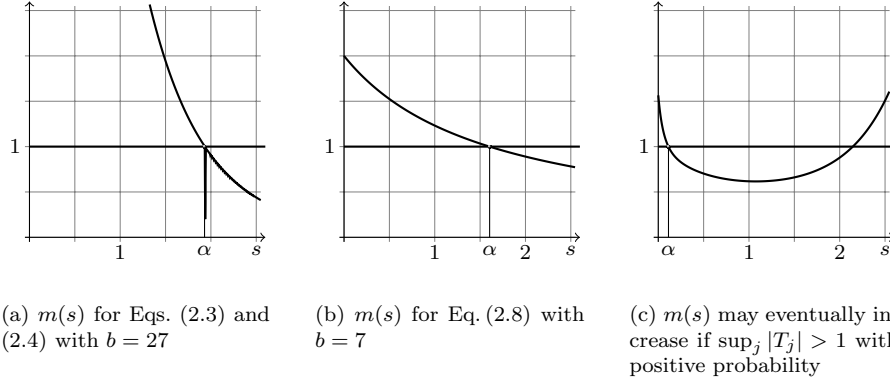


Fig. 1.1: Possible shapes of $s \mapsto m(s)$

While the assumptions imposed up to now are very mild and suffice when $\alpha \neq 1$, it might be not surprising that additional assumptions are needed in the case $\alpha = 1$ in order to overcome severe technical obstacles. We recommend to skip the following part at first reading.

1.3.2 Additional assumptions for the case $\alpha = 1$

We solve (1.1) in the case $\alpha = 1$ under two sets of assumptions, which cover the most relevant cases. Let $\ell = 1 + \dim_{\mathbb{R}} \mathbb{O}(d) = 1 + d(d-1)/2$. Then the first set of additional assumptions for $\alpha = 1$ is

$$\begin{aligned} \mathbb{E} \left[\sum_{j=1}^N \|T_j\|^\alpha \delta_{T_j}(\cdot) \right] \text{ is spread-out (w.r.t. the Haar measure on } \mathbb{S}(d)), \\ \mathbb{E} \left[\sum_{j=1}^N \|T_j\|^\alpha |\log^-(\|T_j\|)|^{\ell+\delta+1} \right] < \infty \text{ and } \mathbb{E}[h_{2\ell+\delta+1}(W_1)] < \infty \quad (\text{A4}) \end{aligned}$$

for some $\delta > 0$ and $h_r(x) := x(\log^+(x))^r \log^+(\log^+(x))$. Notice that in the complex case, for the first condition in (A4) to hold, it is sufficient that, for some $j \in \mathbb{N}$, the law of T_j is spread-out on \mathbb{C} . Let \mathbb{O} denote the smallest closed subgroup of $\mathbb{O}(d)$ which contains the random set $\{T_j/\|T_j\| : j = 1, \dots, N\}$ with probability one. Assumption (A4) will imply that $\mathbb{O} \supseteq \text{SO}(d)$, the subgroup of $\mathbb{O}(d)$ of matrices with determinant 1. In the second set of assumptions, we assume \mathbb{O} to be a finite group:

$$\begin{aligned} \mathbb{O} \text{ is finite, } \mathbb{E} \left[\sum_{j=1}^N \|T_j\|^\alpha \delta_{-\log(\|T_j\|)}(\cdot) \right] \text{ is spread-out,} \\ \mathbb{E} \left[\sum_{j=1}^N \|T_j\|^\alpha (\log^-(\|T_j\|))^2 \right] < \infty \text{ and } \mathbb{E}[h_3(W_1)] < \infty. \quad (\text{A4}') \end{aligned}$$

For a probability measure μ on $\mathbb{O}(d) \times \mathbb{R}$, we say that μ satisfies the minorization condition (M) if there is a nonempty open interval $I \subseteq \mathbb{R}$ and $\gamma > 0$ such that

$$\mu(\text{d}o, \text{d}x) \geq \gamma \mathbb{1}_{\text{SO}(d) \times I}(o, x) H_{\mathbb{O}(d)}(\text{d}o) \text{d}x \quad (\text{M})$$

where $H_{\mathbb{O}(d)}$ is the normalized Haar measure on the compact group $\mathbb{O}(d)$. We will show in Lemma 3.6 that we may assume the following stronger property instead of (A4):

$$(\text{A4}) \text{ holds and } \mathbb{E} \left[\sum_{j=1}^N \|T_j\|^\alpha \delta_{(T_j/\|T_j\|, -\log\|T_j\|)}(\cdot) \right] \text{ satisfies (M).} \quad (\text{A5})$$

1.3.3 Weighted branching

The weighted branching process is a natural tool in the study of equations of the form (1.1). In our context, this process is defined as follows.

Let $\mathbb{V} = \bigcup_{n \in \mathbb{N}_0} \mathbb{N}^n$ denote the infinite Ulam-Harris tree where $\mathbb{N}^0 := \{\emptyset\}$ is the set that contains the empty tuple only. For $u, v \in \mathbb{V}$, $u = (u_1, \dots, u_m)$, $v = (v_1, \dots, v_n)$, we write uv for $(u_1, \dots, u_m, v_1, \dots, v_n)$. We say that v is in generation n , in short: $|v| = n$, if $v \in \mathbb{N}^n$. The restriction of v to its first k components is denoted by $v|_k$.

Let $(\mathbf{C}, \mathbf{T}) := ((C(v), T(v)))_{v \in \mathbb{V}} = ((C(v), T_1(v), T_2(v), \dots))_{v \in \mathbb{V}}$ be a family of i.i.d. copies of the sequence (C, T) . For notational simplicity, we assume $(C(\emptyset), T(\emptyset)) = (C, T)$. Let I_d denote the $d \times d$ identity matrix and let $L(\emptyset) = I_d$. Recursively, for $v \in \mathbb{V}$ and $j \in \mathbb{N}$, we define $L(vj) = L(v)T_j(v)$. Hence, if $v = v_1 \dots v_n \in \mathbb{N}^n$, then

$$L(v) = T_{v_1}(\emptyset) \cdot \dots \cdot T_{v_n}(v|_{n-1}).$$

Notice that the order of multiplication matters. Each of the matrices $T_j(v)$ and $L(v)$ (if nonzero) can be written as the product of a positive scaling factor and an orthogonal matrix:

$$T_j(v) = e^{-S_j(v)} O_j(v) \quad \text{and} \quad L(v) = \|L(v)\| O(v)$$

where

$$\begin{aligned} S_j(v) &= -\log \|T_j(v)\| \in \mathbb{R}, & O_j(v) &= \|T_j(v)\|^{-1} T_j(v) \in \mathbb{O}(d) \\ S(v) &= -\log \|L(v)\| \in \mathbb{R}, & O(v) &= \|L(v)\|^{-1} L(v) \in \mathbb{O}(d) \end{aligned}$$

whenever $\|T_j(v)\| > 0$ or $\|L(v)\| > 0$, respectively. We make the convention that whenever we quantify over the $|v| = n$ as in $\sum_{|v|=n}$ or $\prod_{|v|=n}$, this has to be understood as a quantification over the $|v| = n$ with $\|L(v)\| > 0$ only.

These definitions imply that, for $v = v_1 \dots v_n$, when $\|L(v)\| > 0$,

$$S(v) = \sum_{k=1}^n S_{v_k}(v|_{k-1}) \quad \text{and} \quad O(v) = O_{v_1}(\emptyset) \cdot \dots \cdot O_{v_n}(v|_{n-1}).$$

Here, of course, the order of summation does not matter while the order of matrix multiplication does (in general). We further point out that conditions (A1)–(A2) imply that

$$\lim_{n \rightarrow \infty} \sup_{|v|=n} \|L(v)\| = 0 \quad \text{a.s.} \quad (1.5)$$

(with the convention that $\sup \emptyset = 0$), see [19, Theorem 3] for a reference.

For $u \in \mathbb{V}$ and a function $\Psi = \Psi((\mathbf{C}, \mathbf{T}))$ of the weighted branching process, let $[\Psi]_u$ be defined as $\Psi(((C(uv), T(uv)))_{v \in \mathbb{V}})$, that is, the same function but applied to the weighted branching process rooted at u . The $[\cdot]_u$, $u \in \mathbb{V}$ are called *shift operators*.

1.3.4 Special solutions to smoothing equations

The solutions to (1.1) are connected with the solutions of two related distributional identities, namely, the tilted homogeneous equation for nonnegative random variables

$$W \stackrel{\text{law}}{=} \sum_{j \geq 1} \|T_j\|^\alpha W_j \quad (1.6)$$

where W_1, W_2, \dots are i.i.d. copies of the nonnegative random variable W and independent of T , and the homogeneous equation

$$X \stackrel{\text{law}}{=} \sum_{j \geq 1} T_j X_j \quad (1.7)$$

where X_1, X_2, \dots are i.i.d. copies of X and independent of T . Special solutions to (1.6) and (1.7) can be constructed using the weighted branching process. Indeed, (A2) implies that

$$W_n := \sum_{|v|=n} \|L(v)\|^\alpha = \sum_{|v|=n} e^{-\alpha S(v)}, \quad n \in \mathbb{N}_0 \quad (1.8)$$

defines a nonnegative mean-one martingale. Let $W = \lim_{n \rightarrow \infty} W_n$ a.s. It is known [60] that (A1)–(A3) guarantee $\mathbb{E}[W] = 1$. It can be checked that

$$W = \sum_{|v|=n} \|L(v)\|^\alpha [W]_v \quad \text{a.s.} \quad (1.9)$$

for every $n \in \mathbb{N}_0$. In particular, W is a solution to (1.6). The set of solutions to (1.6) is $\{\text{Law}(cW) : c \geq 0\}$, see [2, 42]. The description of the set of solutions to (1.7) is more delicate and most of the analysis in this paper is concerned with solving it. One aspect of this equation is that special solutions may arise due to balancing effects in the sum $Z_1 := \sum_{j \geq 1} T_j$. This may happen when the matrix $\mathbb{E}[Z_1]$ has eigenvalue 1. Then, for any eigenvector w corresponding to the eigenvalue 1 and with $Z_n := \sum_{|v|=n} L(v)$, the sequence $(Z_n w)_{n \in \mathbb{N}_0}$ defines a martingale. If it converges in probability, we denote its limit by Z^w . Z^w is a function of \mathbf{T} and a solution to the identity

$$Z = \sum_{j \geq 1} T_j [Z]_j \quad \text{a.s.} \quad (1.10)$$

The solutions to (1.10) are described by the following result. Note that $Z = 0$ a.s. always satisfies (1.10), this we call the trivial solution.

Proposition 1.1 *Assume that (A1)–(A3) hold and let Z satisfy (1.10).*

- (a) *If $0 < \alpha < 1$, then $Z = 0$ a.s.*
- (b) *If $\alpha = 1$ and (A4) holds, then $Z = 0$ a.s. If (A4') holds, then a nontrivial solution Z exists iff $w = \mathbb{E}[Z]$ is an eigenvector corresponding to the eigenvalue 1 of $\mathbb{E}[Z_1]$, and then $Z = Ww$.*

- (c) If $1 < \alpha < 2$, then a nontrivial solution Z exists iff $w = \mathbb{E}[Z]$ exists and is an eigenvector corresponding to the eigenvalue 1 of $\mathbb{E}[Z_1]$ and $(Z_n w)_{n \in \mathbb{N}_0}$ is uniformly integrable. If these conditions hold, then $Z = Z^w$ and the sequence $(Z_n w)_{n \in \mathbb{N}_0}$ is bounded in \mathcal{L}^s for all $1 < s < \alpha$.
 If 1 is an eigenvalue of $\mathbb{E}[Z_1]$ associated with the eigenvector w , then a sufficient condition for \mathcal{L}^β -boundedness (and hence uniform integrability) of $(Z_n w)_{n \in \mathbb{N}_0}$ for $\beta \in (\alpha, 2]$ is $\mathbb{E}[|Z_1 w|^\beta] < \infty$ and $m(\beta) < 1$.
- (d) If $\alpha \geq 2$, then a nontrivial solution Z exists iff there is a deterministic $w \neq 0$ such that w is an eigenvector corresponding to the eigenvalue 1 of the random matrix Z_1 a.s., and then $Z = w$ a.s.

Besides W and Z , the class of (U, α) -stable Lévy processes, with U being a closed subgroup of the group of similarity matrices of \mathbb{R}^d , will be relevant for the description of solutions, and is introduced next. This notion is a particular case of $(U, \bar{\alpha})$ -stability, a concept introduced in [23, p. 338].

Let $(Y_t)_{t \geq 0}$ be a Lévy process on \mathbb{R}^d . We say that $(Y_t)_{t \geq 0}$ is (U, α) -stable if there exists a mapping $b : U \rightarrow \mathbb{R}^d$ such that, for all $u \in U$, $t > 0$,

$$uY_t \stackrel{\text{law}}{=} Y_{\|u\|^\alpha t} + tb(u). \quad (1.11)$$

This implies in particular that each Y_t is operator semistable [39], see Section 4.2 for more details. Property (1.11) is equivalent to

$$\Psi(u^\top x) = \|u\|^\alpha \Psi(x) + i\langle x, b(u) \rangle \quad (1.12)$$

for all $u \in U$ for the characteristic exponent Ψ of Y_1 . $(Y_t)_{t \geq 0}$ is said to be *strictly* (U, α) -stable if $b(u) \equiv 0$. We say that a probability measure P on \mathbb{R}^d is (strictly) (U, α) -stable if it is the law of a (strictly) (U, α) -stable Lévy process at time 1.

The general formula of the Lévy measure of a (U, α) -stable law is given in Proposition 1.6 below. Let us just point out that there are no (U, α) -stable Lévy processes for $\alpha > 2$ unless $U \subseteq \mathbb{O}(d)$ and that the larger the group U , the smaller the class of (U, α) -stable Lévy processes. For example, if $U \supseteq \mathbb{R}_> \times \text{SO}(d)$, then $\Psi(x) = -c\|x\|^\alpha$ for some $c \geq 0$, i.e. $(Y_t)_{t \geq 0}$ is strictly α -stable and rotation-invariant. It will be shown (in Remark 3.7 below) that assumption (A4) implies $U \supseteq \mathbb{R}_> \times \text{SO}(d)$. Below, we consider (U, α) -stable Lévy processes, where \mathbb{U} denotes the smallest closed subgroup of the similarity group that covers $\{T_j : j = 1, \dots, N\}$ with probability one.

There are further technicalities to deal with before the main result can be formulated but the complex and homogeneous case, the most important special case in view of applications, is now given as an illustration.

1.3.5 Solutions to complex smoothing equations

In the complex case, each Z_n is a complex random variable, and the condition that $\mathbb{E}[Z_1]$ has eigenvalue 1 is equivalent to $\mathbb{E}[Z_1] = 1$, which in turn is equivalent to $(Z_n)_{n \in \mathbb{N}_0}$ being a (complex) martingale. We write Z for the a.s.

limit of this martingale if it exists, and define $Z = 0$, otherwise. Note that \mathbb{U} is now the smallest closed subgroup of the multiplicative group \mathbb{C}^* that covers $\{T_j : j = 1, \dots, N\}$ with probability one.

Theorem 1.2 *Consider (1.7) in the complex case and assume that (A1)–(A3) hold. Additionally suppose that (A4) or (A4') holds if $\alpha = 1$.*

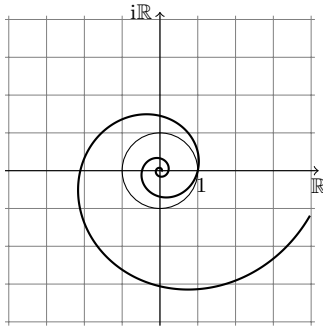
Then a probability distribution on \mathbb{C} is a solution to (1.7) if and only if it is the law of a random variable of the form

$$Y_W + aZ \quad (1.13)$$

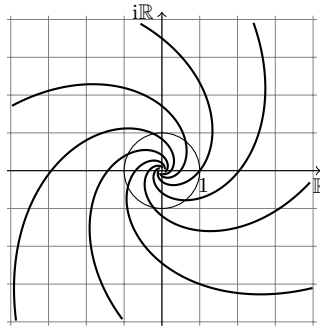
for some $a \in \mathbb{C}$ and a complex strictly (\mathbb{U}, α) -stable Lévy process $(Y)_{t \geq 0}$ independent of (W, Z) .

Theorem 1.2 is a special case of the more general Theorem 1.5. Therefore, we do not give a separate proof of Theorem 1.2.

We have tried to present the result of Theorem 1.2 as concise as possible. This should not hide the fact that it hosts a multitude of different cases. On the one hand, there are several qualitatively different possibilities for the group \mathbb{U} two of which, coming from examples discussed in Section 2.1,



(a) The group \mathbb{U} associated with Eq. (2.4) appearing in the context of b -ary search trees (here, $b = 27$)

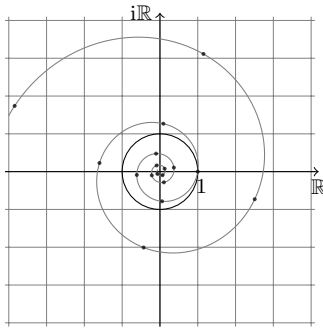
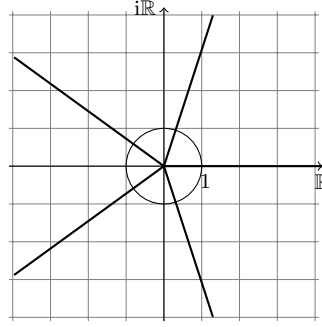


(b) The group \mathbb{U} associated with Eq. (2.8) appearing in the context of cyclic Pólya urns (with $b = 7$)

Fig. 1.2: The group \mathbb{U} in examples

The reader should notice that \mathbb{U} may also be a family of discrete points aligned on a “snail graph” or consist of a finite number of copies of $\mathbb{R}_> := (0, \infty)$ obtained by multiplication with roots of unity as depicted in Figure 1.3. On the other hand, different values of α give rise to qualitatively different regimes. A brief discussion of the implications of Proposition 1.1 and the structure of (\mathbb{U}, α) -stable Lévy processes (described in Proposition 1.6 below) is given in the next remark.

Remark 1.3 Before we discuss the different cases of Theorem 1.2, we point out that (A2) implies that $\mathbb{U} \not\subseteq \mathbb{S}$.

(a) Example where \mathbb{U} is discrete(b) Example where \mathbb{U} consists of circularly arranged copies of $\mathbb{R}_>$ Fig. 1.3: Two possible shapes for \mathbb{U}

- (i) $\alpha < 1$: Then $Z = 0$ a.s., i.e., the second summand in (1.13) vanishes.
- (ii) $\alpha = 1$: Z vanishes unless $\mathbb{E}[Z_1] = 1$. $\mathbb{E}[Z_1] = 1$ and $m(1) = 1$ imply $T_j \in [0, \infty)$ a.s. for all $j \in \mathbb{N}$. In this case, $Z = W$ and $\mathbb{U} \subseteq \mathbb{R}_>$. In fact, Assumptions (A4) and (A4'), resp., imply that $\mathbb{U} = \mathbb{R}_>$.
- (iii) $1 < \alpha < 2$: Z vanishes unless $\mathbb{E}[Z_1] = 1$. In most applications, $\mathbb{E}[Z_1] = 1$ will hold and the full spectrum of solutions given by (1.13) will arise.
- (iv) $\alpha = 2$: In this case, $(Y_t)_{t \geq 0}$ is a centered Gaussian process (or 0). Moreover, $(\mathbb{U}, 2)$ -stability necessitates that real and imaginary part of $(Y_t)_{t \geq 0}$ are i.i.d. centered one-dimensional Brownian motions (or 0) whenever $\mathbb{U} \setminus \mathbb{R} \neq \emptyset$. Further, $Z = 1$ a.s. iff $Z_1 = 1$ a.s. and $Z = 0$, otherwise.
- (v) $\alpha > 2$: Here, $Y_t = 0$ a.s. for all t and $Z = 1$ a.s. iff $Z_1 = 1$ a.s. and $Z = 0$, otherwise. Hence, the set of solutions to (1.7) is either $\{\delta_a : a \in \mathbb{C}\}$ or $\{\delta_0\}$, respectively, where here and throughout the paper, δ_a denotes the Dirac distribution with a point at a .

Remark 1.4 In many cases, the different solutions to (1.13) can be distinguished via their tail behavior. We discuss here in details the most relevant case $1 < \alpha < 2$. In applications, typically $\mathbb{E}[|Z|^\beta] < \infty$ for some $\beta > \alpha$ (see the sufficient condition of Proposition 1.1(c)), whereas Y_W for a non-trivial (\mathbb{U}, α) -stable Lévy process $(Y_t)_{t \geq 0}$ exhibits heavier tails. Indeed, the Lévy measure $\bar{\nu}$ of $(Y_t)_{t \geq 0}$ satisfies $h_1 r^{-\alpha} \leq \bar{\nu}(\{|x| \geq r\}) \leq h_2 r^{-\alpha}$ for $0 < h_1 \leq h_2 < \infty$, which can be derived directly from the (\mathbb{U}, α) -stability and is also implicit in our proofs. According to [72, Corollary 25.8], this implies that, for every $t > 0$, Y_t has all absolute moments of order $< \alpha$ finite, while $\mathbb{E}[|Y_t|^\alpha] = \infty$. Now if $\mathbb{G} := \{\|u\| : u \in \mathbb{U}\} = \mathbb{R}_>$, then, using that $\mathbb{E}[W] = 1 < \infty$ and the independence of W and $(Y_t)_{t \geq 0}$,

$$\mathbb{E}[|Y_W|^p] = \mathbb{E}[(W^{1/\alpha}|Y_1|)^p] = \mathbb{E}[W^{p/\alpha}] \mathbb{E}[|Y_1|^p] < \infty$$

iff $p < \alpha$. In the case where $\mathbb{G} = r^{\mathbb{Z}}$ for some $r > 1$, one can argue similarly using the fact that $\mathbb{E}[\sup_{1 \leq s \leq r} |Y_s|^p] < \infty$ iff $\mathbb{E}[|Y_1|^p] < \infty$, see [72, Theorem 25.18].

More results on the tail behavior of Z (under stronger conditions than imposed here) can be found in [25].

1.3.6 Solutions to multivariate smoothing equations

We continue with the description of the solutions to (1.1) in the general situation. A special solution can be constructed in terms of the weighted branching process. Define

$$W_n^* := \sum_{|v| < n} L(v)C(v), \quad n \in \mathbb{N}_0 \quad (1.14)$$

and let W^* denote the limit in probability as $n \rightarrow \infty$ of W_n^* provided the limit exists. If it exists, W^* satisfies

$$W^* = \sum_{j \geq 1} T_j[W^*]_j + C \quad \text{a.s.} \quad (1.15)$$

and thus constitutes a solution to (1.1). Each of the following is a sufficient condition for the convergence in probability of W_n^* taken from [43, Proposition 2.1], which remains valid in the present context:

- (S1) (A1) and (A2) hold and there is $\beta \in (0, 1]$ with $m(\beta) < 1$ and $\mathbb{E}[|C|^\beta] < \infty$.
- (S2) For some $\beta \geq 1$, $\sup_{n \in \mathbb{N}_0} \mathbb{E}[|W_n^*|^\beta] < \infty$ and either $T_j \geq 0$ for all $j \in \mathbb{N}$ or $\mathbb{E}[C] = 0$.

Notice that if $C = 0$ a.s., then W_n^* converges trivially to $W^* = 0$ a.s.

We now state the main result for the general case.

Theorem 1.5 *Assume that (A1)–(A3) hold and that $W_n^* \rightarrow W^*$ in probability as $n \rightarrow \infty$. If $\alpha = 1$, assume that (A4) or (A4') holds in addition.*

Then a probability distribution on \mathbb{R}^d is a solution to (1.1) if and only if it is the law of a random variable of the form

$$W^* + Y_W + Z \quad (1.16)$$

where Z is a solution of (1.10) and $(Y_t)_{t \geq 0}$ is a strictly (\mathbb{U}, α) -stable Lévy process on \mathbb{R}^d independent of (W^, W, Z) .*

Denote by $E_1 \subseteq \mathbb{R}^d$ the eigenspace corresponding to the eigenvalue 1 of the matrix $\mathbb{E}[Z_1]$ and let $E_1 = \{0\}$ if 1 is not an eigenvalue of $\mathbb{E}[Z_1]$. Then Proposition 1.1 yields that the set of solutions to (1.10) is either empty or parametrized by E_1 . Besides, the solutions are parametrized by the different strictly (\mathbb{U}, α) -stable distributions, which always include $Y_t \equiv 0$, i.e., $W^* + Z$ is also a fixed point. In previous works, the solutions to smoothing equations on the nonnegative halfline [4], on \mathbb{R} [5] and on \mathbb{R}^d in the case where $\mathbb{U} \subseteq \mathbb{R}^*$ [43] have been represented in the form

$$W^* + W^{1/\alpha}Y + Z \quad (1.17)$$

with (W^*, W, Z) defined as here and Y denoting an independent strictly α -stable random variable (where $Y = 0$ is allowed). The same representation is possible here when $\mathbb{R}_> \times \{I_d\} \subseteq \mathbb{U}$ since then $(Y_t)_{t \geq 0}$ is strictly α -stable (or zero) and thus Y_W has the same law as $W^{1/\alpha} Y_1$. In general, solutions to (1.1) do not possess a representation of the form (1.17) as \mathbb{U} need not contain $\mathbb{R}_> \times \{I_d\}$, see e.g. the examples depicted in Figure 1.2.

Observe that W^* and Z are measurable functions of (\mathbf{C}, \mathbf{T}) and satisfy a.s. versions of the inhomogeneous or homogeneous fixed point equation, i.e., (1.15) and (1.10), respectively. Such fixed points are called *endogenous*, a notion coined by Aldous and Bandyopadhyay [1], see [2, Section 6] and [43, Section 3.5] for further information. The fixed point Y_W is not endogenous, for the process $(Y_t)_{t \geq 0}$ introduces additional randomness. Note, however, that W is an endogenous fixed point of the one-dimensional smoothing transform with scalar weights $\|T_1\|^\alpha, \dots, \|T_N\|^\alpha$, see (1.9). The tail behavior of W^* is investigated in [25], and parallels that of Z .

1.4 The class of strictly (U, α) -stable Lévy processes

To complement our main result, we determine the form of the characteristic exponent of strictly (U, α) -stable Lévy processes in this section. The analysis of (U, α) -stable Lévy processes for $U \subseteq \mathbb{O}(d)$ is of no relevance for this paper and simpler than in the case $U \not\subseteq \mathbb{O}(d)$ and therefore omitted here.

Below, denote by ν^α , $0 < \alpha < 2$, a Lévy measure satisfying

$$\nu^\alpha(uB) = \|u\|^{-\alpha} \nu^\alpha(B) \quad (1.18)$$

for all $u \in U$ and all Borel sets $B \subseteq \mathbb{R}^d \setminus \{0\}$. We call such a Lévy measure (U, α) -invariant as well. The structure of such measures is described in Section 4.3. Given ν^α , define the functions

$$\eta_1^\alpha(x) := \frac{1}{|x|^\alpha} \int (1 - \cos(\langle x, y \rangle)) \nu^\alpha(dy), \quad (1.19)$$

$$\eta_2^\alpha(x) := \frac{1}{|x|^\alpha} \int (\sin(\langle x, y \rangle) - \mathbb{1}_{\{\alpha > 1\}} \langle x, y \rangle) \nu^\alpha(dy). \quad (1.20)$$

It is proved in Lemma B.1 that η_i^α are bounded functions, satisfying $\eta_i(u^\top x) = \eta_i(x)$ for all $u \in U$, $x \in \mathbb{R}^d \setminus \{0\}$, $i = 1, 2$.

Now consider the case where the image of U under the homomorphism $U \ni u \mapsto \|u\|$ is $\mathbb{R}_>$. Notice that the groups depicted in Figure 1.2 are of this type. It is proved in [24, Proposition C.1 and Theorem D.13], see also Proposition 4.1, that then $U = \{t^Q : t \in \mathbb{R}_>\} \times C$ for a suitable $d \times d$ -matrix Q and $C := U \cap \mathbb{O}(d)$. Here, $t^Q := e^{(\ln t)Q}$ and we scale Q in such a way that $\|t^Q\| = t$. Observe that (1.11) then implies that any (U, α) -stable law is also operator stable with exponent $\frac{1}{\alpha}Q$, see Section 4.2 for details.

Further, let ρ be a measure on $\mathbb{S}^{d-1} := \{x \in \mathbb{R}^d : |x| = 1\}$ satisfying $\rho(oB) = \rho(B)$ for all $o \in C$ and Borel sets $B \subseteq \mathbb{S}^{d-1}$. We call such a measure C -invariant and define

$$\eta^1(x) := \int_{\mathbb{R}_{>}} \int_{\mathbb{S}^{d-1}} \left(e^{i\langle x, t^Q y \rangle} - 1 - \frac{i\langle x, t^Q y \rangle}{1+t^2} \right) t^{-2} \rho(dy) dt, \quad (1.21)$$

$$(Q - I_d) \gamma^1 := \int_{\mathbb{R}_{>}} \int_{\mathbb{S}^{d-1}} (t^Q y) \frac{2t\langle Qy, y \rangle}{(1+t^2)^2} \rho(dy) dt, \quad (1.22)$$

given the right-hand side in (1.22) is in the range of $(Q - I_d)$.

Below, we write $O := \{u/\|u\| : u \in U\}$ for the projection of U onto $\mathbb{O}(d)$.

Proposition 1.6 *Let $U \not\subseteq \mathbb{O}(d)$ be a closed subgroup of the similarity group and $\alpha > 0$. Then for a d -dimensional Lévy process $(Y_t)_{t \geq 0}$ with characteristic exponent Ψ , the following assertions hold:*

(i) *Let $1 \neq \alpha \in (0, 2)$. $(Y_t)_{t \geq 0}$ is strictly (U, α) -stable iff Ψ is of the form*

$$\Psi(x) = -|x|^\alpha \eta_1^\alpha(x) + i|x|^\alpha \eta_2^\alpha(x) \quad (1.23)$$

for a (U, α) -invariant Lévy measure ν^α .

(ii) *Let $U = \{t^Q : t \in \mathbb{R}_{>}\} \times C$. $(Y_t)_{t \geq 0}$ is strictly $(U, 1)$ -stable iff Ψ is of the form*

$$\Psi(x) = \eta^1(x) + i\langle \gamma^1 + z, x \rangle \quad (1.24)$$

for a C -invariant measure ρ on \mathbb{S}^{d-1} , satisfying $\int \langle x, s \rangle \rho(ds) = 0$ for all x with $Q^\top x = x$ (this also guarantees the existence of γ^1), and any vector z satisfying $uz = \|u\|z$ for all $u \in U$.

(iii) *$(Y_t)_{t \geq 0}$ is strictly $(U, 2)$ -stable iff Ψ is of the form*

$$\Psi(x) = -x^\top \Sigma x/2, \quad x \in \mathbb{R}^d \quad (1.25)$$

for a positive semi-definite symmetric $d \times d$ matrix Σ satisfying $o\Sigma o^\top = \Sigma$ for all $o \in O$.

(iv) *Let $\alpha > 2$. $(Y_t)_{t \geq 0}$ is strictly (U, α) -stable iff $\Psi(x) = 0$ for all $x \in \mathbb{R}^d$, equivalently, $Y_t = 0$ a.s. for all $t \geq 0$.*

Notice that for $\alpha = 1$, the proposition excludes the case where the image of U under the homomorphism $U \ni u \mapsto \|u\|$ is a discrete subgroup of $\mathbb{R}_{>}$. This is because (A4) and (A4'), assumed in the case $\alpha = 1$, imply that $\mathbb{G} = \mathbb{R}_{>}$ and hence the discrete case is of no relevance here.

A description of the matrices Σ satisfying $o\Sigma o^\top = \Sigma$ for all $o \in O$ is given in Proposition 4.4. In particular, if there is no proper subspace $V \subseteq \mathbb{R}^d$, satisfying $oV = V$ for all $o \in O$, then Σ is a scalar multiple of the identity matrix.

1.5 Further organization of the paper

In Section 2, we apply our main results to the examples mentioned in the introduction as well as to further important models. The rest of the paper is then devoted to the proofs of the main results, with the major, probabilistic part of the proof being given in Section 3, at the beginning of which we will also introduce further notation and concepts relevant for the proofs. More algebraic considerations, concerned with the structure of (U, α) -stable Lévy processes, are contained in Section 4. The appendix contains a Choquet-Deny lemma for functions on U and a rate-of-convergence result for Markov renewal processes; the latter result will be needed only in the case $\alpha = 1$.

2 Applications of the main results

In this section, we discuss the examples from the introduction, as well as further applications of our main results to the study of Biggins' martingale with complex parameter or Gaussian multiplicative chaos.

2.1 Applications of Theorem 1.2

2.1.1 b -ary search trees

b -ary search trees are b -ary trees which are basic data structures in computer science used in searching and sorting, see [62] for the definition and background information. Each node of a b -ary search tree can store up to $b - 1$ elements from a set of distinct real numbers x_1, \dots, x_n , called the set of keys. For the problems considered, it constitutes no loss of generality to assume $\{x_1, \dots, x_n\} = \{1, \dots, n\}$. Denote by Y_n the space requirement of a b -ary search tree under the random permutation model, i.e., (x_1, \dots, x_n) is a uniform permutation of the set $\{1, \dots, n\}$ and Y_n is the number of nodes in the resulting b -ary search tree.

Let $b \geq 4$. The asymptotic behavior of Y_n is coded in the equation

$$\chi(z) := \prod_{j=1}^{b-1} (z + j) - b! = 0, \quad z \in \mathbb{C}. \quad (2.1)$$

The root with largest absolute value is $\lambda_1 = 1$. Let λ_2 denote the root with second-largest real part and $\text{Im}(\lambda_2) > 0$. Then $\text{Re}(\lambda_2) \in (0, 1)$. If $\text{Re}(\lambda_2) \leq 1/2$, equivalently, $b \leq 26$, then after centering and norming, Y_n is asymptotically normal, see [58]. By using martingale methods, Chauvin and Pouyanne [30] have shown that if $\text{Re}(\lambda_2) > 1/2$, equivalently, $b \geq 27$, then

$$Y_n = \text{const} \cdot n + 2\text{Re}(n^{\lambda_2} X) + o(n^{\text{Re}(\lambda_2)}) \quad (2.2)$$

where $o(n^{\text{Re}(\lambda_2)})$ is a term that, after dividing by $n^{\text{Re}(\lambda_2)}$, tends to 0 a.s. and in \mathcal{L}^2 , and X is a complex-valued random variable. Since n^{λ_2} is complex,

$n^{-\operatorname{Re}(\lambda_2)}(Y_n - \text{const} \cdot n)$ does not converge in distribution, but behaves like $2\operatorname{Re}(n^{\operatorname{Im}(\lambda_2)}X)$.

Fill and Kapur [37] showed that X solves the smoothing equation

$$X \stackrel{\text{law}}{=} \sum_{j=1}^b V_j^{\lambda_2} X_j \quad (2.3)$$

where V_1, \dots, V_b are the spacings of $b-1$ i.i.d. random variables uniformly distributed over $(0, 1)$, U_1, \dots, U_{b-1} , say. In other words, $V_j = U_{(j)} - U_{(j-1)}$ for $j = 1, \dots, b$ with $U_{(0)} = 0$, $U_{(b)} = 1$ and $(U_{(1)}, \dots, U_{(b-1)})$ being the order statistics of (U_1, \dots, U_{b-1}) . What is more, Fill and Kapur showed that (the law of) X is the unique solution to (2.3) subject to the additional constraints $\mathbb{E}[X] = \mu$ and $\mathbb{E}[|X|^2] < \infty$ where $\mu \neq 0$ is a given complex constant.

Later, using an embedding into continuous-time multi-type Markov branching processes, Chauvin et al. [29] established a connection between the law of X and the complex smoothing equation

$$X \stackrel{\text{law}}{=} e^{-\lambda_2 T} (X_1 + \dots + X_b) \quad (2.4)$$

where T has the same distribution as the sum $\tau_1 + \dots + \tau_{b-1}$ of independent random variables with τ_j being exponentially distributed with parameter j , $j = 1, \dots, b-1$. Again, the connection concerns the solution to (2.4) with fixed expectation $\mathbb{E}[X] = \mu \neq 0$ and finite second moment $\mathbb{E}[|X|^2] < \infty$.

Here we will consider Eq. (2.3) only; the study of (2.4) bears a striking similarity. One can check that V_1, \dots, V_b are identically distributed with Lebesgue density $(b-1)(1-x)^{b-2} \mathbb{1}_{(0,1)}(x)$. Hence, for $z \in \mathbb{C}$,

$$\begin{aligned} \mathbb{E}[V_1^z + \dots + V_b^z] &= b(b-1) \int_0^1 x^z (1-x)^{b-2} dx = \frac{b! \Gamma(z+1)}{\Gamma(z+b)} \\ &= \frac{b!}{(z+1) \cdot \dots \cdot (z+b-1)} \end{aligned} \quad (2.5)$$

where Γ denotes Euler's gamma function. We conclude that in the present context the function $s \mapsto m(s)$ defined in (1.4) takes the form

$$m(s) = b! \prod_{j=1}^{b-1} \frac{1}{\operatorname{Re}(\lambda_2)s + j}, \quad (2.6)$$

Figure 1.1a shows the graph of $s \mapsto m(s)$. Notice that $m(\alpha) = 1$ only for $\alpha = 1/\operatorname{Re}(\lambda_2)$. In particular, the phase transition at $b = 26$ is visible here since $\alpha \geq 2$ for $b \leq 26$ and $\alpha \in (1, 2)$ for $b \geq 27$, and the transition between normal and stable behavior occurs at $\alpha = 2$. Further, from (2.5) for $z = \lambda_2$, we conclude that $\mathbb{E}[Z_1] = 1$ for $Z_1 = V_1^{\lambda_2} + \dots + V_b^{\lambda_2}$. It follows from (2.6) that $m(s) < 1$ for all $s > \alpha$ and it can be checked that $\mathbb{E}[|Z_1|^2] < \infty$. Thus, the sufficient condition of Proposition 1.1(c) applies and $Z_n \rightarrow Z$ a.s. and in \mathcal{L}^2 for a complex random variable Z with $\mathbb{P}(Z \neq 0) > 0$.

By Theorem 1.2, the set of solutions to (2.3) is given by all laws of random variables of the form

$$Y_W + aZ \quad (2.7)$$

where $(Y_t)_{t \geq 0}$ is a strictly (\mathbb{U}, α) -stable Lévy process independent of (W, Z) . Here, $\mathbb{U} = \{e^{\lambda_{2t}} : t \in \mathbb{R}\}$. For $b = 27$, this group is depicted in Figure 1.2a. The solution of interest can be singled out by moment properties using Remark 1.4 and is $X = \mu Z$. This gives in particular a positive answer to the question posed in [29, Remark 4.5] about the existence of further solutions with infinite second moment.

2.1.2 Cyclic Pólya urns

Consider an urn containing finitely many balls of b different types, $1, \dots, b$. At each step, a ball is drawn and placed back into the urn together with an assortment of new balls, the types of which depend on the type of the ball drawn. Such a scheme is called a generalized Pólya urn. If the replacement rule is such that if a ball of type k is drawn, then it is placed back into the urn together with a ball of type $k + 1$ if $k < b$ and of type 1 if $k = b$, the urn is called cyclic.

Let $R_{n,k}$ be the number of balls of type 1 in a cyclic urn after n steps when starting with exactly one ball of type k and no other ball. We have $\mathbb{E}[R_{n,k}] = \frac{n}{b} + O(1)$ as $n \rightarrow \infty$, see e.g. [53, Lemma 6.7].

If $b \leq 6$, then $R_{n,k} - \frac{n}{b}$, suitably scaled, is asymptotically normal. If $b \geq 7$, let $\zeta = \exp(2\pi i/b) = \xi + i\eta$ be a primitive b th root of unity. It has been shown with martingale methods [46, 69] and via the contraction method [53, Section 6.3] that $n^{-\xi}(R_{n,k} - \frac{n}{b})$ has an asymptotic periodic behavior (similar to Eq. (2.2)) that is governed by the law of a random variable X which is the unique non-degenerate solution with expectation $2/(b\Gamma(\zeta + 1))$ and finite second moment of the equation

$$X \stackrel{\text{law}}{=} U^\zeta X_1 + \zeta(1 - U)^\zeta X_2 \quad (2.8)$$

where X_1, X_2 are i.i.d. copies of X that are independent of U which has the uniform distribution on $[0, 1]$.

Therefore,

$$m(s) = \mathbb{E}[|U^\zeta|^s + |\zeta(1 - U)^\zeta|^s] = \frac{2}{1 + \xi s},$$

see Figure 1.1b for a plot of $s \mapsto m(s)$. Thus $\alpha = 1/\xi$ and $\alpha \in (1, 2)$ iff $\xi = \cos(2\pi/b) > \frac{1}{2}$ iff $b \geq 7$. In particular, the phase transition at $b = 6$ is visible here as the phase transition between normal and stable behavior occurs at $\alpha = 2$.

(A1)–(A3) are readily checked to be valid in the present context. Further,

$$\mathbb{E}[Z_1] = \mathbb{E}[U^\zeta + \zeta(1 - U)^\zeta] = (1 + \zeta)\mathbb{E}[U^\zeta] = 1.$$

Since $m(2) < 1$ and $\mathbb{E}[|Z_1|^2] \leq 4 < \infty$, the sufficient condition in Proposition 1.1(c) is fulfilled and we conclude that $Z_n \rightarrow Z$ a.s. and in \mathcal{L}^2 as $n \rightarrow \infty$ for a random variable Z with $\mathbb{E}[Z] = 1$ and $\mathbb{E}[|Z|^2] < \infty$. We have $\mathbb{U} = \{\zeta^k e^{\zeta t} : 0 \leq k < b, t \in \mathbb{R}\}$; Figure 1.2b is a depiction of \mathbb{U} in the case $b = 7$. It follows that the whole spectrum of solutions given in (1.13) appears. The special solution X appearing in the description of the limiting behavior of $R_{n,k} - \frac{n}{b}$ is the unique solution to (2.8) with mean $2/(b\Gamma(\zeta + 1))$ and finite variance. Since $\mathbb{E}[|Y_W|^2] = \infty$ for any non-trivial (\mathbb{U}, α) -stable Lévy process $(Y_t)_{t \geq 0}$ by Remark 1.4, it is $X = 2Z/(b\Gamma(\zeta + 1))$.

2.1.3 Asymptotic size of fragmentation trees

In Kolmogorov's conservative fragmentation model [15, 54] an object of mass $x = 1$, say, is split into b parts with respective masses $0 \leq V_1, \dots, V_b < 1$ where $b \geq 2$ is a fixed integer and V_1, \dots, V_b are random variables with $V_1 + \dots + V_b = 1$ a.s. The splitting procedure is repeated with the resulting objects using independent copies of the splitting vector (V_1, \dots, V_b) to determine the relative sizes of the emerging objects. Janson and Neininger [47] investigated the size $N(\epsilon)$ of the random fragmentation tree the vertices of which correspond to all objects created in the fragmentation process that have mass strictly $\geq \epsilon$ for some given $\epsilon > 0$. They showed that the asymptotics of $N(\epsilon)$ are coded in the function ψ that maps $z \in \mathbb{C}$ to $\mathbb{E}[\sum_{j=1}^b V_j^z]$ (whenever the expectation exists). To be more precise, denote by $1 = \lambda_1, \lambda_2, \lambda_3, \dots$ the roots of the equation $\psi(z) = 1$ with the convention that $1 = \operatorname{Re}(\lambda_1) > \operatorname{Re}(\lambda_2) \geq \operatorname{Re}(\lambda_3) \geq \dots$. Then, under suitable assumptions, when $\operatorname{Re}(\lambda_2) \leq \frac{1}{2}$, $N(\epsilon)$ suitably shifted and scaled, converges in distribution to a centered normal. On the other hand, when $\operatorname{Re}(\lambda_2) > \frac{1}{2}$, $N(\epsilon)$ exhibits a periodic limiting behavior governed by a complex-valued random variable X , with finite second moment and a fixed expectation $\gamma \in \mathbb{C}$, satisfying the distributional equation

$$X \stackrel{\text{law}}{=} \sum_{j=1}^b V_j^{\lambda_2} X_j \quad (2.9)$$

where X_1, \dots, X_b are i.i.d. copies of X and independent of (V_1, \dots, V_b) .

This equation is in the scope of our analysis. Indeed, letting $T_j := V_j^{\lambda_2}$ for $j = 1, \dots, b$, and $T_j = 0$ for $j > b$, we have

$$m(s) = \mathbb{E}\left[\sum_{j \geq 1} |T_j|^s\right] = \mathbb{E}\left[\sum_{j=1}^b V_j^{\operatorname{Re}(\lambda_2)s}\right] = \psi(\operatorname{Re}(\lambda_2)s). \quad (2.10)$$

In particular, $m(\alpha) = 1$ iff $\alpha = 1/\operatorname{Re}(\lambda_2)$. Again, the phase transition between normal and stable fluctuations is reflected in the equation since $\alpha < 2$ iff $\operatorname{Re}(\lambda_2) > \frac{1}{2}$. Further, the conditions (A1)–(A3) are easily checked to hold. Since $m(2) < 1$ and

$$\mathbb{E}\left[\left|\sum_{j=1}^b V_j^{\lambda_2}\right|^2\right] \leq b \mathbb{E}\left[\sum_{j=1}^b V_j^{2\operatorname{Re}(\lambda_2)}\right] < 1,$$

the sufficient condition of Proposition 1.1(c) is fulfilled and hence the martingale $(Z_n)_{n \in \mathbb{N}_0}$ converges a.s. and in \mathcal{L}^2 and the special solution X used to describe the limiting behavior of $N(\epsilon)$ is γZ . The general form of solutions is given by (1.13).

It is assumed in [47] that each V_j has an absolutely continuous component, hence we are in the continuous case; in fact, $\mathbb{U} = \{e^{\lambda_2 t} : t \in \mathbb{R}\}$. If the values of λ_2 are the same, we obtain the same class of (\mathbb{U}, α) -stable Lévy processes as in the case of b -ary search trees, but the law of W will depend on the explicit distribution of V_1, \dots, V_b , not only on their support.

2.1.4 Biggins' martingale with complex parameter and complex Gaussian multiplicative chaos

Eq. (1.10) arises naturally in the context of branching random walks: Consider an initial ancestor at the origin with children placed on \mathbb{R} according to a point process \mathcal{Z} on \mathbb{R} with $\mathbb{E}[\mathcal{Z}(\mathbb{R})] > 1$ (supercritical case). Each child produces offspring with positions relative to its location given by an independent copy of \mathcal{Z} , and so on. Denoting by $(\mathcal{S}(v))_{|v|=n}$ the positions of the n th generation particles, consider the Laplace transform with complex parameter λ of the random point measure formed by the n th generation particles,

$$\mathcal{M}_n(\lambda) := \sum_{|v|=n} e^{-\lambda \mathcal{S}(v)}.$$

If $\mathbf{m}(\lambda) := \mathbb{E}[\sum_{|v|=1} e^{-\lambda \mathcal{S}(v)}]$ is finite, then $\mathbb{E}[\mathcal{M}_n(\lambda)] = \mathbf{m}(\lambda)^n$, and

$$\mathcal{W}_n(\lambda) := \frac{\mathcal{M}_n(\lambda)}{\mathbf{m}(\lambda)^n}$$

is a complex-valued martingale with $\mathbb{E}[\mathcal{W}_1(\lambda)] = 1$, called *Biggins' martingale*, see [17]. Sufficient conditions for the convergence of $\mathcal{W}_n(\lambda)$ to a nondegenerate limit $\mathcal{W}(\lambda)$ are studied in [18]. Upon defining $T_j := e^{-\lambda \mathcal{S}(j)} / \mathbf{m}(\lambda)$ and using the same shift notation as for the weighted branching process, one obtains that

$$\mathcal{W}(\lambda) = \sum_{j \geq 1} \frac{e^{-\lambda \mathcal{S}(j)}}{\mathbf{m}(\lambda)} [\mathcal{W}(\lambda)]_j = \sum_{j \geq 1} T_j [\mathcal{W}(\lambda)]_j \quad \text{a.s.}$$

Thus, $\mathcal{W}(\lambda)$ is a solution to (1.10), in particular, $\mathcal{W}_n(\lambda) = Z_n$ in our notation.

The sufficient conditions for the \mathcal{L}^β -convergence of $\mathcal{W}_n(\lambda)$ from [18, Theorem 1] translate as follows: (2.1) there is equivalent to $\mathbb{E}[(\sum_{j \geq 1} |T_j|)^\gamma] < \infty$ for some $\gamma \in (1, 2]$, while (2.2) equals $m(\beta) < 1$ for some $\beta \in (1, \gamma]$. These imply the sufficient conditions of Proposition 1.1(c).

It is an important open problem to find equivalent conditions for the convergence of Z_n to a nondegenerate limit, for it may also provide educated guesses in the theory of complex Gaussian multiplicative chaos. This is the complex analogue of real Gaussian multiplicative chaos, which was introduced by Kahane [49], see also [70] for a recent review and more details. Complex

Gaussian multiplicative chaos is a complex random measure $\mathcal{M}^{\gamma,\beta}$ (with parameters $\beta, \gamma > 0$) on \mathbb{R} , which is obtained via a limiting procedure from *regularized* measures $\mathcal{M}_\epsilon^{\gamma,\beta}$. The question is about the correct renormalization needed to obtain convergence. Three phases (Phases I, II and III) appear, see Figure 1 in [56]. The suitable scaling can be guessed from the behavior of Biggins' martingale with complex parameter. A particular instance, which was studied by Madaule, Rhodes and Vargas [61] in order to provide intuition for the behavior on the boundary between phases I/II ($\gamma \in (1/2, 1)$, $\gamma + \beta = 1$), is

$$\mathcal{M}_n(\gamma, \beta) := \sum_{|v|=n} \exp(-\gamma \mathcal{S}(v) + i\beta \sqrt{2 \ln 2} \mathcal{S}'(v)),$$

where $(\mathcal{S}(v))_{v \in \mathbb{V}}$, $(\mathcal{S}'(v))_{v \in \mathbb{V}}$ are independent branching random walks with binary branching and i.i.d. displacements with normal laws with mean $2 \log 2$ and variance $2 \log 2$ for $\mathcal{S}(v)$ resp. mean 0 and variance 1 for $\mathcal{S}'(v)$.

As described above, if $\mathcal{W}_n := (\mathbb{E}[\mathcal{M}_n])^{-1} \mathcal{M}_n$ converges to a limit \mathcal{W} , then this limit is a solution to a smoothing equation, and equal to the particular solution Z . In the setting of [61], one has $\mathbb{E}[\mathcal{M}_1(\gamma, \beta)] = 1$ and $m(s) = \exp((s\gamma - 1)^2 \log 2)$, hence $\alpha = 1/\gamma \in (1, 2)$. The phase transition is reflected in the fact that $m'(\alpha) = 0$ on the boundary between phases I/II. In this case, the sufficient conditions of Proposition 1.1 do not hold, nevertheless it is proved in [61, Theorem 1] that \mathcal{W}_n converges to a nontrivial limit.

2.2 Application of Theorem 1.5

We end this section with the study of the example described in Section 1.2.2. In contrast to the examples above, the relevant solution will be given by Y_W .

Bassetti and Matthes [14, Section 6.2] study the equation

$$V \stackrel{\text{law}}{=} LV_1 + RV_2, \quad (2.11)$$

where V, V_1, V_2 are i.i.d. random vectors in \mathbb{R}^3 , independent of the random pair (L, R) of similarities which satisfies

$$m(2) = \mathbb{E}[\|L\|^2 + \|R\|^2] = 1 \text{ and } m(p) = \mathbb{E}[\|L\|^p + \|R\|^p] < 1$$

for some $p \in (2, 3)$. Thus (A1)–(A3) are satisfied with $\alpha = 2$, and all solutions to (2.11) are given by Theorem 1.5. Since the equation is homogeneous, W^* vanishes, while Y_1 can be any centered multivariate normal random variable with a covariance matrix that is invariant under conjugation by elements of \mathbb{O} .

Bassetti and Matthes further assume that $L = lA$ for independent random variables $l \in \mathbb{R}$ and $A \in \mathbb{O}(d)$, and that the law of $A^\top x$ dominates the volume measure on \mathbb{S}^{d-1} for every $x \in \mathbb{S}^{d-1}$. This implies that \mathbb{O} acts transitively on \mathbb{S}^{d-1} and hence that scalar multiples of $\Sigma = I_d$ are the only possible choices for the covariance matrix of Y_1 . As Proposition 4.4 below shows, the

weaker assumption that there is no \mathbb{U} -invariant proper subspace readily implies $\Sigma = I_d$.

Finally, $Z = Z^w \neq 0$ if and only if $Lw + Rw = w$ a.s., which corresponds to the conservation of momentum. In [14], only centered solutions were considered, the physical interpretation of which is that the centre of gravity does not move. Here, we see that the centre of gravity may have drift Z^w , which corresponds to the validity of Newton's first law in this context.

In recent papers by Dolera and Regazzini [34] and Bassetti et. al. [13], solutions of the Boltzmann equation for Maxwellian molecules in \mathbb{R}^3 have been studied directly, rather than its simplifications like the Kac caricature. These steady state solutions cannot be written directly as solutions to smoothing equations, but the techniques employed seem to be very similar. We hope that our results allow for a better understanding; in particular under which conditions rotation invariant solutions appear.

3 Proofs of the main results

In this section, we prove our main results: Proposition 1.1, Theorem 1.5 and Proposition 1.6. At the beginning, we collect the relevant notation and introduce tools and concepts which are used in the proofs below.

3.1 Notation

3.1.1 Vector spaces, sets, matrices, etc

We work in the d -dimensional space \mathbb{R}^d . We think of an element $x \in \mathbb{R}^d$ as a column vector. We write x^\top for the corresponding row vector. e_1, \dots, e_d denote the canonical basis vectors of \mathbb{R}^d . By $\langle \cdot, \cdot \rangle$, we denote the standard Euclidean scalar product on \mathbb{R}^d , that is, $\langle x, y \rangle = x^\top y$ for $x, y \in \mathbb{R}^d$. We write $|x|$ for $\sqrt{\langle x, x \rangle}$, the Euclidean norm of x . For a set $B \subseteq \mathbb{R}^d$, ∂B denotes the boundary of B , $B^\perp = \{y \in \mathbb{R}^d : \langle x, y \rangle = 0 \text{ for all } x \in B\}$ denotes the orthogonal complement of B in \mathbb{R}^d . $B_r := \{x \in \mathbb{R}^d : |x| < r\}$ denotes the ball of radius r centered around the origin, $r > 0$. For a given real $d \times d$ matrix A , we write A_{ij} for the coefficient in the i th row and the j th column of A and $\|A\| := \sup_{|x|=1} |Ax|$ for its norm. A^\top , $\text{tr}(A)$ and $\det(A)$ denote the transpose, the trace and the determinant of A , respectively. For $\lambda \in \mathbb{R}$, we set $E_\lambda(A) = \{x \in \mathbb{R}^d : Ax = \lambda x\}$. We write I_k for the $k \times k$ identity matrix, $k \in \mathbb{N}$.

3.1.2 Probability spaces, expectations, etc

Throughout the paper, we fix a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ which is large enough to carry all random variables appearing in the paper. By $\mathbb{E}[\cdot]$ and $\text{Var}[\cdot]$

we denote expectation resp. variance with respect to \mathbb{P} . We also consider expectations of random vectors and random matrices, which are defined componentwise. For a random vector Y , we write $\text{Cov}[Y] := \mathbb{E}[(Y - \mathbb{E}[Y])(Y^\top - \mathbb{E}[Y]^\top)]$ for the covariance matrix of Y with respect to \mathbb{P} . $\mathbb{1}_A$ denotes the indicator function of a set A and we write $\mathbb{E}[Y; A]$ for $\mathbb{E}[Y \mathbb{1}_A]$. Further, $\text{Cov}[Y; A]$ is the covariance matrix of the random vector $Y \mathbb{1}_A$.

3.1.3 Relevant groups

The following groups are of relevance. $\mathbb{S}(d) \subseteq GL(d, \mathbb{R})$ is the group of similarity matrix, i.e., scalar multiples of orthogonal matrices. \mathbb{U} denotes the smallest closed subgroup of $\mathbb{S}(d)$ that covers $\{T_j : j = 1, \dots, N\}$ with probability one. In the complex case, we identify \mathbb{U} with a subgroup of the multiplicative group \mathbb{C}^* of \mathbb{C} . Similarly, \mathbb{O} is the smallest closed subgroup of $\mathbb{O}(d)$ which contains the random set $\{T_j / \|T_j\| : j = 1, \dots, N\}$ with probability one. In the complex case, we identify \mathbb{O} with the smallest multiplicative subgroup of $\mathbb{S} = \{z \in \mathbb{C} : |z| = 1\}$ which contains the random set $\{T_j / |T_j| : j = 1, \dots, N\}$ with probability one. In this case, either $\mathbb{O} = \{e^{2\pi i k/m} : k = 0, \dots, m-1\}$ for some $m \in \mathbb{N}$ or $\mathbb{O} = \mathbb{S}$. Finally, let \mathbb{G} be the smallest closed multiplicative subgroup of $\mathbb{R}_>$ that covers the random set $\{\|T_j\| : j = 1, \dots, N\}$ with probability one. Equivalently, \mathbb{G} is the image of the group \mathbb{U} under the homomorphism $u \mapsto \|u\|$. There are three possibilities: (C) $\mathbb{G} = \mathbb{R}_>$; (G) $\mathbb{G} = r^{\mathbb{Z}}$ for some $r > 1$; (T) $\mathbb{G} = \{1\}$. The trivial case (T) is excluded by (A3). We refer to (G) as the geometric or r -geometric case and to (C) as the continuous or non-geometric case.

More information about the structure of \mathbb{U} is provided in Section 4.1. In particular, $\mathbb{U} = A_{\mathbb{U}} \rtimes C_{\mathbb{U}}$ for a one-parameter group $A_{\mathbb{U}}$ which is isomorphic to \mathbb{G} , and $C_{\mathbb{U}} = \mathbb{U} \cap \mathbb{O}(d)$. Note that in general, $C_{\mathbb{U}} \subsetneq \mathbb{O}$.

Throughout the paper, we call a measure μ on \mathbb{R}^d G -(left)-invariant for a group G of matrices if $\mu(g^{-1}B) = \mu(B)$ for all $g \in G$ and all Borel sets $B \subseteq \mathbb{R}^d$.

3.1.4 Weighted branching

We set $\mathcal{F}_n := \sigma((C(v), T(v)) : |v| < n)$ and $\mathcal{F} := \sigma(\mathcal{F}_n : n \in \mathbb{N}_0)$. (\mathbf{C}, \mathbf{T}) serves as an abbreviation for the family $((C(v), T(v)))_{v \in \mathbb{V}}$. We further assume that on the basic probability space, a family $\mathbf{X} := (X_v)_{v \in \mathbb{V}}$ of i.i.d. random variables is defined which is independent of \mathcal{F} . We do not specify the law of $X := X_\emptyset$ here, but typically, X will be a solution to (1.1) or (1.7).

3.2 Characteristic functions

It is natural to approach Equation (1.1) via characteristic functions. Indeed, the distributional equation (1.1) for an \mathbb{R}^d -valued random variable X is equiv-

alent to the functional equation

$$\phi(x) = \mathbb{E} \left[e^{i\langle x, C \rangle} \prod_{j \geq 1} \phi(T_j^\top x) \right], \quad x \in \mathbb{R}^d \quad (3.1)$$

for the characteristic function $\phi(x) = \mathbb{E}[e^{i\langle x, X \rangle}]$ of X . In the homogeneous case, the functional equation takes the simpler form

$$\phi(x) = \mathbb{E} \left[\prod_{j \geq 1} \phi(T_j^\top x) \right], \quad x \in \mathbb{R}^d. \quad (3.2)$$

Solving (1.1) is equivalent to finding all characteristic functions ϕ of \mathbb{R}^d -valued random variables which satisfy (3.1).

3.3 Proof of Theorem 1.5: The direct inclusion

We are now ready to prove the direct inclusion of Theorem 1.5. Here, with the direct inclusion, we mean the assertion that any distribution of a random variable of the form (1.16) is a solution to (1.1).

Proof (of the direct inclusion of Theorem 1.5) Consider the situation of Theorem 1.5 and let $X = W^* + Y_W + Z$ where $(Y_t)_{t \geq 0}$ is a strictly (\mathbb{U}, α) -stable Lévy process independent of the family (\mathbf{C}, \mathbf{T}) , in particular independent of (W^*, W, Z) . Denote the characteristic function of X by ϕ and the characteristic exponent of Y_1 by Ψ , that is, $\mathbb{E}[e^{i\langle x, Y_t \rangle}] = \exp(t\Psi(x))$. By assumption, Ψ satisfies (1.12) with $b(u) \equiv 0$ for $u \in \mathbb{U}$. Using the independence of (W^*, W, Z) and $(Y_t)_{t \geq 0}$, we conclude that

$$\phi(x) = \mathbb{E} \left[\exp(i\langle x, W^* \rangle + i\langle x, Z \rangle + W\Psi(x)) \right], \quad x \in \mathbb{R}^d. \quad (3.3)$$

As $([W^*]_j, [W]_j, [Z]_j)$ is a copy of (W^*, W, Z) , (3.3) still holds when (W^*, W, Z) is replaced by $([W^*]_j, [W]_j, [Z]_j)$, $j \in \mathbb{N}$. Furthermore, since $([W^*]_j, [W]_j, [Z]_j)$ is independent of \mathcal{F}_1 , we conclude that

$$\begin{aligned} \phi(T_j^\top x) &= \mathbb{E} \left[\exp(i\langle T_j^\top x, [W^*]_j \rangle + i\langle T_j^\top x, [Z]_j \rangle + [W]_j \Psi(T_j^\top x)) \mid \mathcal{F}_1 \right] \\ &= \mathbb{E} \left[\exp(i\langle x, T_j [W^*]_j \rangle + i\langle x, T_j [Z]_j \rangle + \|T_j\|^\alpha [W]_j \Psi(x)) \mid \mathcal{F}_1 \right] \end{aligned}$$

a.s. for every $x \in \mathbb{R}^d$, where in the last step we have used (1.12) and the fact that $T_j \in \mathbb{U}$ a.s. Consequently, for every $x \in \mathbb{R}^d$, we infer

$$\begin{aligned}
& \mathbb{E} \left[e^{i\langle x, C \rangle} \prod_{j \geq 1} \phi(T_j^\top x) \right] \\
&= \mathbb{E} \left[e^{i\langle x, C \rangle} \prod_{j \geq 1} \mathbb{E} \left[e^{i\langle x, T_j[W^*]_j \rangle + i\langle x, T_j[Z]_j \rangle + \|T_j\|^\alpha [W]_j \Psi(x)} \mid \mathcal{F}_1 \right] \right] \\
&= \mathbb{E} \left[e^{i\langle x, C \rangle + \sum_{j \geq 1} (i\langle x, T_j[W^*]_j \rangle + i\langle x, T_j[Z]_j \rangle + \|T_j\|^\alpha [W]_j \Psi(x))} \right] \\
&= \mathbb{E} \left[e^{i\langle x, C + \sum_{j \geq 1} T_j[W^*]_j \rangle + i\langle x, \sum_{j \geq 1} T_j[Z]_j \rangle + \sum_{j \geq 1} \|T_j\|^\alpha [W]_j \Psi(x)} \right] \\
&= \mathbb{E} \left[e^{i\langle x, W^* \rangle + i\langle x, Z \rangle + W \Psi(x)} \right] = \phi(x),
\end{aligned}$$

i.e., ϕ solves (3.1). Hence, X is a solution to (1.1). \square

Most of the remainder of this paper is devoted to the proof of the converse inclusion of Theorem 1.5 and the description of the class of strictly (\mathbb{U}, α) -stable Lévy processes. We begin with a short section on a common technique in the theory of branching processes, an exponential change of measure.

3.4 Exponential change of measure

Recall from Section 1.3.3 the definition of the weighted branching process. We define the associated random walk $(L_n)_{n \in \mathbb{N}_0}$ on $\mathbb{S}(d)$ by the many-to-one formula

$$\mathbb{E}[f(L_0, \dots, L_n)] = \mathbb{E} \left[\sum_{|v|=n} \|L(v)\|^\alpha f((L(v|_k))_{k=0, \dots, n}) \right] \quad (3.4)$$

for all nonnegative Borel-measurable functions $f : \mathbb{S}(d)^{n+1} \rightarrow \mathbb{R}_{\geq 0}$. (A2) implies that the law of $(L_n)_{n \in \mathbb{N}_0}$ is a proper probability measure. From this definition, it can be checked that $(L_n)_{n \in \mathbb{N}_0}$ is a multiplicative random walk on the group $\mathbb{U} \subseteq \mathbb{S}(d)$. We define $S_n := -\log \|L_n\|$ and $O_n := L_n / \|L_n\|$. $(S_n)_{n \in \mathbb{N}_0}$ is a standard random walk on \mathbb{R} , while $(O_n)_{n \in \mathbb{N}_0}$ is a multiplicative random walk on $\mathbb{O}(d)$. The step distribution of $(S_n)_{n \in \mathbb{N}_0}$ is given by

$$\mathbb{P}(S_1 \in \cdot) = \mathbb{E} \left[\sum_{j=1}^N \|T_j\|^\alpha \delta_{-\log \|T_j\|}(\cdot) \right]. \quad (3.5)$$

Consequently, when (A3) holds, $\mathbb{E}[S_1] = -m'(\alpha) \in (0, \infty)$.

Later on, we will use that (3.4) remains valid under certain stopping rules: Considering $\tau(t) := \inf\{n \in \mathbb{N}_0 : S_n > t \text{ and } S_k \leq t \text{ for all } k < n\}$, (3.4) gives

$$\begin{aligned}
& \mathbb{E}[f(L_0, \dots, L_n) \mathbb{1}_{\{\tau(t)=n\}}] \\
&= \mathbb{E} \left[\sum_{|v|=n} \|L(v)\|^\alpha f((L(v|_k))_{k=0, \dots, n}) \mathbb{1}_{\{S(v) > t \geq S(v|_k) \forall k < n\}} \right].
\end{aligned}$$

Summing over all $n \in \mathbb{N}_0$ and defining the *coming generation at time $t \geq 0$* ,

$$\mathcal{C}(t) = \{v \in \mathbb{V} : \|L(v)\| > 0 \text{ and } S(v) > t \geq S(v|_k) \text{ for all } k < |v|\}, \quad (3.6)$$

we infer that in particular

$$\mathbb{E}[f(S_{\tau(t)-1}, S_{\tau(t)}, O_{\tau(t)})] = \mathbb{E}\left[\sum_{v \in \mathcal{C}(t)} e^{-\alpha S(v)} f(S(v|_{|v|-1}), S(v), O(v))\right] \quad (3.7)$$

for all nonnegative Borel-measurable functions $f : \mathbb{R}^2 \times \mathbb{O}(d) \rightarrow \mathbb{R}_{\geq}$. See [55] for more information on stopping lines and further references.

3.5 Multiplicative martingales

Let X be a solution to (1.1) and denote the characteristic function of X by ϕ . We will show that ϕ is the characteristic function of a random variable of the form (1.16). As in previous works on fixed points of smoothing transformations [2, 4, 5, 20, 21, 43], we make use of multiplicative martingales. As this technique is well-known by now, we keep the presentation short here and refer to the above references for more detailed expositions. For $x \in \mathbb{R}^d$, define

$$M_n(x) := \exp(i\langle x, W_n^* \rangle) \cdot \prod_{|v|=n} \phi(L(v)^T x), \quad n \in \mathbb{N}_0. \quad (3.8)$$

The fact that ϕ solves (3.1) implies that $(M_n(x))_{n \in \mathbb{N}_0}$ is a complex-valued martingale. Since it is bounded by 1 in absolute value, it converges a.s. and in mean. We denote its a.s. limit by $M(x)$ and note that

$$\phi(x) = \mathbb{E}[M(x)], \quad x \in \mathbb{R}^d. \quad (3.9)$$

In order to determine $\phi(x)$, it thus suffices to determine $M(x)$.

We begin with a key lemma. We will sometimes write $M(x) = M(\omega, x)$ in order to make more transparent, whether we consider $M(x)$ as a random variable, or $x \mapsto M(\omega, x)$ as a function of x .

Lemma 3.1 *There is an \mathcal{F} -measurable set N with $\mathbb{P}(N) = 0$ such that, for $\omega \in N^c$, $M_n(\omega, \cdot) \rightarrow M(\omega, \cdot)$ pointwise on \mathbb{R}^d . Further, $x \mapsto M(\omega, x)$ is the characteristic function of a probability distribution on \mathbb{R}^d for all $\omega \in N^c$.*

An obvious modification of the proof of Theorem 1 in [26] yields the result. We refrain from giving any details.

Solving the inhomogeneous equation (1.1) can be reduced to solving the associated homogeneous equation (1.7) along the lines of [5, Section 5], a sketch of the reduction argument will be given in Section 3.11 below. For now, we restrict our attention to the homogeneous case and assume that $C = 0$ a.s. Then (3.8) takes the simpler form

$$M_n(x) = \prod_{|v|=n} \phi(L(v)^T x), \quad n \in \mathbb{N}_0. \quad (3.10)$$

We claim that for all ω from a set of probability one, $M(\omega, \cdot)$ is the characteristic function of an infinitely divisible law. Since $\sup_{|v|=n} \|L(v)\| \rightarrow 0$ a.s., we can assume without loss of generality that the set N from Lemma 3.1 is such that $N^c \subseteq \{\sup_{|v|=n} \|L(v)\| \rightarrow 0\}$. Now pick an $\omega \in N^c$. We view $M_n(\omega, \cdot)$ as the characteristic function of $\sum_{|v|=n} L(v)X_v$ conditional given $(L(v))_{v \in \mathbb{V}}$. Thus, $M(\omega, \cdot)$ is the limit of characteristic functions of the row sums in a triangular array which is independent and infinitesimal. Hence, it is the characteristic function of an infinitely divisible law and we can write $M(x) = \exp(\Psi(x))$ for a random characteristic exponent $\Psi(x)$ which is of the form

$$\Psi(x) = i\langle W', x \rangle - \frac{x^\top \Sigma x}{2} + \int \left(e^{i\langle x, y \rangle} - 1 - i\langle x, y \rangle \mathbb{1}_{[0,1]}(|y|) \right) \nu(dy) \quad (3.11)$$

for $W' = W'(\mathbf{T}) \in \mathbb{R}^d$, a covariance matrix $\Sigma = \Sigma(\mathbf{T})$ and a Lévy measure $\nu = \nu(\mathbf{T})$ on \mathbb{R}^d (see [52, p. 290]).

That W', Σ and ν are indeed functions of \mathbf{T} comes from the following representation, valid for triangular arrays (see [52, Chapter 15]): Using the convention $\int_{\{h < |x| \leq 1\}} = -\int_{\{1 < |x| \leq h\}}$ when $h > 1$, it holds on N^c ,

$$W' = W^h + \int_{\{h < |x| \leq 1\}} x \nu(dx) \quad (3.12)$$

where W^h is defined by

$$W^h := \lim_{n \rightarrow \infty} \sum_{|v|=n} \mathbb{E}[L(v)X_v; |L(v)X_v| \leq h \mid \mathcal{F}] \quad (3.13)$$

for every $h > 0$ with $\nu(\{|x| = h\}) = 0$,

$$\Sigma = \Sigma^h - \int_{\{h < |x| \leq 1\}} xx^\top \nu(dx) \quad (3.14)$$

where Σ^h is defined by

$$\Sigma^h := \lim_{n \rightarrow \infty} \sum_{|v|=n} \text{Cov}[L(v)X_v; |L(v)X_v| \leq h \mid \mathcal{F}], \quad (3.15)$$

and

$$\int f(x) \nu(dx) = \lim_{n \rightarrow \infty} \sum_{|v|=n} \int f(L(v)x) F(dx) \quad (3.16)$$

for all continuous functions f with compact support on $\overline{\mathbb{R}^d} \setminus \{0\}$ ($\overline{\mathbb{R}^d}$ denotes the one-point compactification of \mathbb{R}^d). Eq. (3.16) also yields that ν is a random measure (see [51, Lemma 4.1]), i.e., the mapping $\mathbf{T} \mapsto \int f(x) \nu(\mathbf{T}, dx)$ is measurable for every nonnegative Borel measurable function f on $\mathbb{R}^d \setminus \{0\}$. In order to show that W' and Σ are random variables as well, we need some more preparation (the problem is to choose h in a measurable way).

Right now, we can use that W' , Σ and ν are functions of \mathbf{T} and apply the shift operator $[\cdot]_v$. Arguing as in the proof of Lemma 4.3 in [5], we conclude that on N^c

$$M(x) = \prod_{|v|=n} [M]_v(L(v)^\top x) \quad \text{for all } x \in \mathbb{R}^d \text{ and } n \in \mathbb{N}_0. \quad (3.17)$$

Now using (3.17) in (3.11), we conclude that for all $n \in \mathbb{N}_0$, on N^c

$$\begin{aligned} \Psi(x) &= i \sum_{|v|=n} \langle [W']_v, L(v)^\top x \rangle - \sum_{|v|=n} \frac{x^\top L(v) [\Sigma]_v L(v)^\top x}{2} \\ &\quad + \sum_{|v|=n} \int \left(e^{i\langle L(v)^\top x, y \rangle} - 1 - i\langle L(v)^\top x, y \rangle \mathbb{1}_{[0,1]}(|y|) \right) [\nu]_v(dy) \\ &= i \left\langle \sum_{|v|=n} L(v) [W']_v, x \right\rangle - \frac{1}{2} x^\top \sum_{|v|=n} L(v) [\Sigma]_v L(v)^\top x \\ &\quad + \sum_{|v|=n} \int \left(e^{i\langle x, L(v)y \rangle} - 1 - i\langle x, L(v)y \rangle \mathbb{1}_{[0,1]}(|L(v)y|) \right) [\nu]_v(dy) \\ &\quad - \sum_{|v|=n} \int \left(i\langle x, L(v)y \rangle \mathbb{1}_{[0,1]}(|y|) - i\langle x, L(v)y \rangle \mathbb{1}_{[0,1]}(|L(v)y|) \right) [\nu]_v(dy). \end{aligned} \quad (3.18)$$

Since the last term contributes to the random shift, using the uniqueness of the Lévy triplet, we get:

$$\int f(y) \nu(dy) = \sum_{|v|=n} \int f(L(v)y) [\nu]_v(dy) \quad \text{on } N^c \quad (3.19)$$

$$\text{and} \quad \Sigma = \sum_{|v|=n} L(v) [\Sigma]_v L(v)^\top \quad \text{on } N^c \quad (3.20)$$

for all $n \in \mathbb{N}_0$ and all nonnegative Borel-measurable functions f on $\overline{\mathbb{R}^d} \setminus \{0\}$. We will use (3.19) and (3.20) to determine ν and Σ , respectively.

The next lemma gives an important estimate for ν and will allow us to infer the measurability of W' and Σ .

Lemma 3.2 *There is a multiplicatively r -periodic (if $\mathbb{G} = r^{\mathbb{Z}}$) or constant (if $\mathbb{G} = \mathbb{R}_{>}$) function $\mathfrak{h} : [0, \infty) \rightarrow (0, \infty)$ such that $t \mapsto \mathfrak{h}(t)t^\alpha$ is nondecreasing and such that on N^c ,*

$$\nu(B_{|x|}^c) = W \mathfrak{h}(|x|^{-1}) |x|^{-\alpha} \quad \text{for all } x \in \mathbb{R}^d \setminus \{0\}. \quad (3.21)$$

Proof Define $f : [0, \infty) \rightarrow [0, 1]$ by

$$f(t) := \begin{cases} \mathbb{E}[\exp(-\nu(B_{t-1}^c))] & \text{for } t > 0, \\ 1 & \text{for } t = 0. \end{cases}$$

f is decreasing in t and continuous at 0 since $B_{t-1}^c \downarrow \emptyset$ as $t \downarrow 0$. The most important property of f , however, is that it solves the functional equation of the smoothing transformation as studied in [2]. Indeed, using (3.19) and the fact that $L(v) = \|L(v)\| O(v)$ for an orthogonal matrix $O(v)$ and $O(v)^{-1} B_r^c = B_r^c$ for all $r > 0$, we get

$$\begin{aligned} f(t) &= \mathbb{E} \left[\exp \left(- \sum_{|v|=n} [\nu]_v (L(v)^{-1} B_{t-1}^c) \right) \right] \\ &= \mathbb{E} \left[\prod_{|v|=n} \exp \left(- [\nu]_v B_{(\|L(v)\|t)^{-1}}^c \right) \right] = \mathbb{E} \left[\prod_{|v|=n} f(\|L(v)\|t) \right]. \end{aligned}$$

Consider the limit M_f of the multiplicative martingale associated with f , i.e., $M_f(t) = \lim_{n \rightarrow \infty} \prod_{|v|=n} f(\|L(v)\|t)$, $t \geq 0$. By Theorem 8.3 in [2], $M_f(t) = \exp(-W\mathfrak{h}(t)t^\alpha)$ a.s. for all $t \geq 0$ and some function \mathfrak{h} with properties as above. By the arguments given in the proof of [5, Lemma 4.8],

$$\nu(B_{|x|}^c) = W\mathfrak{h}(|x|^{-1})|x|^{-\alpha} \quad \text{for all } x \in \mathbb{R}^d \setminus \{0\} \text{ on } N^c. \quad (3.22)$$

Since (3.21) holds, we can pick some $0 < h \leq 1$ such that \mathfrak{h} is continuous at h , in particular, $\nu(\partial B_h) = 0$ on N^c . Using this h in (3.12), (3.13), (3.14) and (3.15) implies the asserted measurability statements for W' and Σ .

We call a Lévy triplet (W', Σ, ν) with these measurability properties an \mathcal{F} -measurable Lévy triplet. We summarize the results of the above discussion in the following proposition.

Proposition 3.3 *Assume that (A1) and (A2) hold. Let X be a solution to (1.7) and denote its characteristic function and distribution function by ϕ and F , respectively. Denote by $(M_n(x))_{n \in \mathbb{N}_0}$, $x \in \mathbb{R}^d$ the multiplicative martingales associated with X and defined by (3.10). Then, on an \mathcal{F} -measurable set N^c with $\mathbb{P}(N) = 0$,*

$$M_n(x) \rightarrow M(x) \quad \text{as } n \rightarrow \infty \quad (3.23)$$

for all $x \in \mathbb{R}^d$. For $\omega \in N^c$, $x \mapsto M(\omega, x)$ is a characteristic function and possesses a representation $M(x) = \exp(\Psi(x))$ for all $x \in \mathbb{R}^d$ where Ψ is given by (3.11). (W', Σ, ν) is an \mathcal{F} -measurable random Lévy triplet satisfying (3.12)–(3.16).

Next, we state some consequences of Proposition 3.3 concerning the tail behavior of solutions to (1.7). These will be useful when determining ν and Σ .

3.6 Tail estimates

If X is a solution to (1.7), then

$$\limsup_{t \rightarrow \infty} t^\alpha \mathbb{P}(|X| > t) < \infty. \quad (3.24)$$

If, additionally, the random Lévy measure ν of the limit of the multiplicative martingale M associated with X (or its characteristic function ϕ) vanishes a.s., then the stronger estimate

$$\limsup_{t \rightarrow \infty} t^\alpha \mathbb{P}(|X| > t) = 0 \quad (3.25)$$

holds. The derivation of these estimates can be carried out along the lines of [43, Lemma 4.7] and [5, Lemma 4.9], we refrain from giving more details here. As a consequence of (3.24), we obtain the following inequality for

$$L_\beta(t) := \mathbb{E}[|X|^\beta; |X| \leq t] \quad (3.26)$$

with $\beta > \alpha$:

$$\begin{aligned} \limsup_{t \rightarrow \infty} t^{\alpha-\beta} L_\beta(t) &\leq \limsup_{t \rightarrow \infty} t^{\alpha-\beta} \int_0^t \beta x^{\beta-1} \mathbb{P}(|X| > x) dx \\ &\leq C \limsup_{t \rightarrow \infty} t^{\alpha-\beta} \int_0^t x^{\beta-\alpha-1} dx < \infty \end{aligned} \quad (3.27)$$

where C is a constant depending on α, β and the law of X (via (3.24)). In the case where the random Lévy measure ν vanishes a.s., using (3.25) instead of (3.24) produces the stronger estimate

$$\lim_{t \rightarrow \infty} t^{\alpha-\beta} L_\beta(t) = 0. \quad (3.28)$$

3.7 Determining ν

The characterization of ν is given by the following lemma.

Lemma 3.4 *Assume that (A1)–(A3) are in force and let ν be a \mathcal{F} -measurable random Lévy measure. Then ν satisfies (3.19) if and only if $\nu = W\bar{\nu}$ a.s. for a deterministic (\mathbb{U}, α) -invariant Lévy measure $\bar{\nu}$, i.e., satisfying*

$$\bar{\nu}(gB) = \|g\|^{-\alpha} \bar{\nu}(B) \quad (3.29)$$

for all $g \in \mathbb{U}$ and all Borel sets $B \subseteq \mathbb{R}^d \setminus \{0\}$. Further, $\bar{\nu} = 0$ if $\alpha \geq 2$.

Proof We first prove the sufficiency. To this end, suppose $\nu = W\bar{\nu}$ a.s. for a deterministic Lévy measure $\bar{\nu}$ satisfying (3.29). It suffices to check validity of (3.19) when f is the indicator function of an arbitrary Borel set $B \subseteq \mathbb{R}^d \setminus \{0\}$. Using (3.29) and then (1.9), we obtain

$$\begin{aligned} \sum_{|v|=n} [\nu]_v(L(v)^{-1}B) &= \sum_{|v|=n} [W]_v \bar{\nu}(L(v)^{-1}B) \\ &= \sum_{|v|=n} \|L(v)\|^\alpha [W]_v \bar{\nu}(B) = W\bar{\nu}(B) = \nu(B) \quad \text{a.s.} \end{aligned}$$

For the converse implication, assume that ν is an \mathcal{F} -measurable random Lévy measure satisfying (3.19).

The idea of the proof is as follows: We define $\bar{\nu} := \mathbb{E}[\nu]$ and prove that $g \mapsto \|g\|^{-\alpha} \bar{\nu}(g^{-1}B)$ is a constant function in g (this necessitates proving that $(\omega, g) \mapsto \nu(g^{-1}B)$ is product-measurable). Then we show that the identity $\nu(B) = W\bar{\nu}(B)$ a.s. holds simultaneously for enough sets B (namely, a countable generator of the Borel σ -field which is closed under finite intersections) on a common \mathbb{P} -null set.

Step 1: We start by proving the measurability statement: For every Borel set $B \subseteq \mathbb{R}^d \setminus \{0\}$, the mapping $(\omega, g) \mapsto \nu(g^{-1}B)$ from $\Omega \times \mathbb{S}(d)$ to \mathbb{R}_{\geq} is $\mathcal{F} \otimes \mathfrak{B}(\mathbb{S}(d))$ -measurable where $\mathfrak{B}(\mathbb{S}(d))$ denotes the Borel σ -field on $\mathbb{S}(d)$. Let $B \subseteq \mathbb{R}^d \setminus \{0\}$ be closed, $B^{1/k} := \{y \in \mathbb{R}^d : |y-z| < \frac{1}{k} \text{ for some } z \in B\}$ and f_k be a continuous function satisfying $0 \leq f_k \leq 1$, $f_k|_B = 1$ and $f_k|_{(B^{1/k})^c} = 0$ (such a function exists, for instance, by Urysohn's lemma). Then $f_k \rightarrow \mathbb{1}_B$ as $k \rightarrow \infty$. By (3.16),

$$\int f_k(gx) \nu(dx) = \lim_{n \rightarrow \infty} \sum_{|v|=n} \int f_k(L(v)gx) F(dx)$$

By Fubini's theorem, the right-hand side is $\mathcal{F} \otimes \mathfrak{B}(\mathbb{S}(d))$ -measurable and by the dominated convergence theorem, the left-hand side tends to $\nu(g^{-1}B)$ as $k \rightarrow \infty$. Hence $(\omega, g) \mapsto \nu(g^{-1}B)$ is $\mathcal{F} \otimes \mathfrak{B}(\mathbb{S}(d))$ -measurable. For every fixed $r > 0$, this extends to the Dynkin system generated by all closed sets $B \subseteq B_r^c$, hence to all Borel measurable sets $B \subseteq \mathbb{R}^d \setminus \{0\}$.

Step 2: Now we introduce the sets that will form the countable generator. We define, for $x \in \mathbb{R}^d$ and $\epsilon > 0$,

$$I_x^\epsilon = \left\{ y \in \mathbb{R}^d : |y| \geq |x|, \left| \frac{y}{|y|} - \frac{x}{|x|} \right| \leq \epsilon \right\}.$$

We will determine $\nu(I_{x/|x|^2}^\epsilon)$. First notice that, for $o \in \mathbb{O}(d)$,

$$\left| \frac{oy}{|oy|} - \frac{x}{|x|} \right| = \left| o^{-1} \left(o \frac{y}{|y|} - \frac{x}{|x|} \right) \right| = \left| \frac{y}{|y|} - o^{-1} \frac{x}{|x|} \right|$$

and, therefore, for a similarity $g = \|g\| o$ with $\|g\| > 0$,

$$g^{-1}I_x^\epsilon = \left\{ y \in \mathbb{R}^d : |y| \geq \frac{|x|}{\|g\|}, \left| \frac{y}{|y|} - o^{-1} \frac{x}{|x|} \right| \leq \epsilon \right\} = I_{g^{-1}x}^\epsilon = I_{g^\top x / \|g\|^2}^\epsilon \quad (3.30)$$

where in the last step, we have used that $g^{-1} = \|g\|^{-1} o^\top = g^\top / \|g\|^2$. Using this in (3.19) with $f = \mathbb{1}_{I_{x/|x|^2}^\epsilon}$ gives

$$\nu(I_{x/|x|^2}^\epsilon) = \sum_{|v|=n} [\nu]_v (L(v)^{-1} I_{x/|x|^2}^\epsilon) = \sum_{|v|=n} [\nu]_v (I_{L(v)^\top x / |L(v)^\top x|^2}^\epsilon) \text{ a.s.} \quad (3.31)$$

Step 3: For $x \in \mathbb{R}^d \setminus \{0\}$, define $\Psi_\epsilon(x) := \nu(I_{x/|x|^2}^\epsilon)|x|^{-\alpha}$. By (3.21), $I_{x/|x|^2}^\epsilon \subseteq B_{|x|^{-1}}^c$ implies

$$\nu(I_{x/|x|^2}^\epsilon) \leq \nu(B_{|x|^{-1}}^c) = W\mathfrak{h}(|x|)|x|^\alpha \quad \text{a.s.} \quad (3.32)$$

Consequently, with $h^* := \sup_{t>0} \mathfrak{h}(t)$, we have

$$\sup_{x \in \mathbb{R}^d \setminus \{0\}} \Psi_\epsilon(x) \leq Wh^* \quad \text{a.s.} \quad (3.33)$$

Further, by (3.31),

$$\Psi_\epsilon(x) = \sum_{|v|=n} \|L(v)\|^\alpha [\Psi_\epsilon]_v(L(v)^\top x) \quad \text{for all } x \in \mathbb{R}^d \setminus \{0\} \text{ a.s.} \quad (3.34)$$

For $x \in \mathbb{R}^d \setminus \{0\}$, define $\psi_{\epsilon,x} : \mathbb{U} \rightarrow \mathbb{R}_>$ via

$$\psi_{\epsilon,x}(g) = \mathbb{E}[\Psi_\epsilon(g^\top x)], \quad g \in \mathbb{U}.$$

$\psi_{\epsilon,x}$ is measurable (since $(\omega, g) \mapsto \nu(g^{-1}B)$ is product-measurable), nonnegative and bounded (due to (3.33) and the fact that $\mathbb{E}[W] = 1$). Further, as a consequence of (3.34) and the change of measure (3.4), we have

$$\begin{aligned} \psi_{\epsilon,x}(g) &= \mathbb{E}[\Psi_\epsilon(g^\top x)] = \mathbb{E}\left[\sum_{|v|=n} \|L(v)\|^\alpha [\Psi_\epsilon]_v(L(v)^\top g^\top x)\right] \\ &= \mathbb{E}\left[\sum_{|v|=n} \|L(v)\|^\alpha \psi_{\epsilon,x}(gL(v))\right] = \mathbb{E}[\psi_{\epsilon,x}(gL_n)] \end{aligned}$$

for all $g \in \mathbb{U}$. In other words, $\psi_{\epsilon,x}$ solves a Choquet-Deny functional equation on the (possibly non-Abelian) topological group \mathbb{U} . From the Choquet-Deny lemma (see Lemma A.1 in the appendix) we infer that $\psi_{\epsilon,x}$ is $\mathbb{P}(L_n \in \cdot)$ -a.s. constant on \mathbb{U} . Consequently, there is a constant $c_{\epsilon,x} \geq 0$ such that, with $U = \{\psi_{\epsilon,x} = c_{\epsilon,x}\}$, $\mathbb{P}(L_n \in U) = 1$ for all $n \in \mathbb{N}_0$. Notice that this implies

$$\mathbb{E}\left[\sum_{|v|=n} \|L(v)\|^\alpha \mathbb{1}_U(L(v))\right] = \mathbb{P}(L_n \in U) = 1$$

and hence $\mathbb{P}(L(v) \in U \text{ for all } |v| = n \text{ with } \|L(v)\| > 0) = 1$. This together with the martingale convergence theorem and (3.34) yields

$$\begin{aligned} \Psi_\epsilon(x) &= \lim_{n \rightarrow \infty} \mathbb{E}[\Psi_\epsilon(x) | \mathcal{F}_n] = \lim_{n \rightarrow \infty} \sum_{|v|=n} \|L(v)\|^\alpha \mathbb{E}[[\Psi_\epsilon]_v(L(v)^\top x) | \mathcal{F}_n] \\ &= \lim_{n \rightarrow \infty} \sum_{|v|=n} \|L(v)\|^\alpha \psi_{\epsilon,x}(L(v)) = Wc_{\epsilon,x} \quad \text{a.s.} \end{aligned}$$

In terms of $\nu(I_{x/|x|^2}^\epsilon)$, this reads

$$\nu(I_{x/|x|^2}^\epsilon) = Wc_{\epsilon,x}|x|^\alpha \quad \text{a.s.}$$

Setting $\bar{\nu}(\cdot) := \mathbb{E}[\nu(\cdot)]$ and taking expectations in the above equation gives $\bar{\nu}(I_{x/|x|^2}^\epsilon) = c_{\epsilon,x}|x|^\alpha$ and hence

$$\nu(I_{x/|x|^2}^\epsilon) = W\bar{\nu}(I_{x/|x|^2}^\epsilon) \quad \text{a.s.} \quad (3.35)$$

Step 4: Using (3.30), we conclude that, for every $g \in \mathbb{U}$,

$$\nu(g^{-1}I_{x/|x|^2}^\epsilon) = \nu(I_{g^\top x/|g^\top x|^2}^\epsilon) = W\psi_{\epsilon,x}(g^\top)|g^\top x|^\alpha = \|g\|^\alpha \nu(I_{x/|x|^2}^\epsilon) \quad \text{a.s.}$$

and hence, upon taking expectation, we arrive at the transformation formula (3.29) for $\bar{\nu}$, valid for all $g \in \mathbb{U}$ and sets $I_{x/|x|^2}^\epsilon$. (3.29) extends to finite intersections of the form $I = I_{x_1/|x_1|^2}^{\epsilon_1} \cap \dots \cap I_{x_n/|x_n|^2}^{\epsilon_n}$ for $\mathbf{x} = (x_1, \dots, x_n) \in (\mathbb{R}^d \setminus \{0\})^n$ and $\epsilon = (\epsilon_1, \dots, \epsilon_n) \in \mathbb{R}_{>}^n$. To be more precise, first notice that in such an intersection, one can assume that $|x_1| = \dots = |x_n|$ (otherwise, let $r := \min_{j=1}^n |x_j|$ and notice that I does not change when x_j is replaced by $rx_j/|x_j|$, $j = 1, \dots, n$). (3.32) then carries over with $I_{x/|x|^2}^\epsilon$ replaced by I and $|x|$ replaced by $|x_1|$. The obvious counterparts for I of (3.30) and (3.31) hold true and one can define $\Psi_\epsilon(\mathbf{x}) := \nu(I)|x_1|^{-\alpha}$. The remaining arguments apply almost without changes and give

$$\nu(I) = W\bar{\nu}(I) \quad \text{a.s.} \quad (3.36)$$

and the counterpart of (3.29). Now notice that the finite intersections of sets $I_{x/|x|^2}^\epsilon$ with $x \in \mathbb{Q}^d \setminus \{0\}$ and rational $\epsilon > 0$ form a countable generator of the Borel σ -field on \mathbb{R}^d and on a set of probability one, $\nu(I) = W\bar{\nu}(I)$ for all sets from this generator. It follows by an application of the uniqueness theorem from measure theory that $\nu(B) = W\bar{\nu}(B)$ for all Borel sets $B \subseteq \mathbb{R}^d \setminus \{0\}$. Analogously, (3.29) extends to all Borel measurable $B \subseteq \mathbb{R}^d \setminus \{0\}$.

Step 5: It remains to show that $\bar{\nu}$ is a Lévy measure and that $\bar{\nu} = 0$ if $\alpha \geq 2$. For the former, notice that $\bar{\nu}$ is a measure on $\mathbb{R}^d \setminus \{0\}$ and that

$$\int (|x|^2 \wedge 1) \bar{\nu}(dx) < \infty \quad \text{iff} \quad \int (|x|^2 \wedge 1) \nu(dx) < \infty \quad \text{a.s.}$$

by (3.36). For the proof of the second claim, notice that taking expectations in (3.21) gives $\bar{\nu}(B_t^c) = \mathfrak{h}(t^{-1})t^{-\alpha}$ for all $t > 0$ for a multiplicatively \mathbb{G} -periodic function \mathfrak{h} such that $\mathfrak{h}(t)t^\alpha$ is nondecreasing in t . Because of these properties we have $h_* := \inf_{t>0} \mathfrak{h}(t) = 0$ iff $h \equiv 0$. Now assume that $\alpha \geq 2$. We show that then $h_* = 0$. Indeed, since $\bar{\nu}$ is a Lévy measure, integration by parts gives

$$\begin{aligned} \infty &> \int_{\{|x|^2 \leq 1\}} |x|^2 \bar{\nu}(dx) = \int_0^1 2t\bar{\nu}(B_t^c \setminus B_1^c) dt \\ &= \int_0^1 2t\mathfrak{h}(t^{-1})t^{-\alpha} dt - \mathfrak{h}(1) \geq 2h_* \int_0^1 t^{1-\alpha} dt - \mathfrak{h}(1) \end{aligned}$$

which implies that $h_* = 0$. □

Any Lévy measure $\bar{\nu}$ satisfying the invariance property (3.29) can be factorized into a radial and spherical part, similar to the decomposition valid for Lévy measures of stable laws. The details are given in Proposition 4.3 in Section 4 below.

3.8 Determining Σ

Now we use the identity

$$\Sigma = \sum_{|v|=n} L(v)[\Sigma]_v L(v)^\top \quad \text{a.s.}, \quad (3.20)$$

to determine Σ .

Lemma 3.5 *Assume that (A1)–(A3) hold and let Σ be a \mathcal{F} -measurable random covariance matrix. Then Σ satisfies (3.20) if and only if $\Sigma = W\Sigma$ a.s. for a deterministic covariance matrix Σ satisfying*

$$\Sigma = o\Sigma o^\top \quad (3.37)$$

for all $o \in \mathbb{O}$. Further, $\Sigma = 0$ if $\alpha \neq 2$.

Proof Our first observation is that $\text{tr}(\Sigma)$, the trace of Σ , is a nonnegative endogenous fixed point of the one-dimensional smoothing transform with weights $(\|T_j\|^2)_{j \geq 1}$. To see this, notice that it follows from (3.20) and the fact that the trace is invariant under conjugations with orthogonal transformations that

$$\begin{aligned} \text{tr}(\Sigma) &= \text{tr} \left(\sum_{|v|=n} L(v)[\Sigma]_v L(v)^\top \right) = \sum_{|v|=n} \|L(v)\|^2 \text{tr}(O(v)[\Sigma]_v O(v)^\top) \\ &= \sum_{|v|=n} \|L(v)\|^2 [\text{tr}(\Sigma)]_v \quad \text{a.s.} \end{aligned}$$

From Theorem 6.2 in [2], we conclude that $\text{tr}(\Sigma) = cW$ for some constant $c \geq 0$ if $\alpha = 2$, and $\text{tr}(\Sigma) = 0$ a.s., otherwise.

Now recall that all eigenvalues of a covariance matrix, i.e., a positive semi-definite, symmetric matrix are nonnegative and that there is a basis of \mathbb{R}^d consisting only of eigenvectors of that matrix. Further, the trace of the matrix equals the sum of its eigenvalues.

In the case $\alpha \neq 2$, this implies that all eigenvalues of Σ equal 0 a.s., which means that the matrix itself vanishes a.s. This, in turn, proves the assertion in the case where $\alpha \neq 2$.

In the case $\alpha = 2$, we can conclude that $\text{tr}(\Sigma)$ has finite expectation since $\mathbb{E}[W] = 1$. This implies that Σ_{ij} is integrable for $i, j = 1, \dots, d$ since

$$|\Sigma_{ij}| = |e_i \Sigma e_j| \leq \|\Sigma\| \leq \text{tr}(\Sigma)$$

where for the last inequality, we have used that the norm of Σ is the largest eigenvalue of Σ while the trace of Σ is the sum of its eigenvalues. Consequently,

$\Sigma := \mathbb{E}[\Sigma]$ is a symmetric $d \times d$ matrix. We claim that $\Sigma = o\Sigma o^\top$ for all $o \in \mathbb{O}$. To prove this, notice that since Σ is symmetric and positive semi-definite, it has $n \leq d$ distinct real eigenvalues $\lambda_1 > \dots > \lambda_n \geq 0$ such that $d_1 + \dots + d_n = d$ where d_j denotes the dimension of the eigenspace $E_{\lambda_j}(\Sigma)$ corresponding to the eigenvalue λ_j , $j = 1, \dots, n$. For arbitrary $x \in E_{\lambda_1}(\Sigma)$, first apply (3.20) for $n = 1$ and then condition w.r.t. \mathcal{F}_1 to obtain

$$\lambda_1 |x|^2 = x^\top \Sigma x = \mathbb{E} \left[\sum_{j=1}^N \|T_j\|^2 x^\top O_j \Sigma O_j^\top x \right]. \quad (3.38)$$

On the other hand, $x^\top O_j \Sigma O_j^\top x \leq \lambda_1 |x|^2$ for all $j = 1, \dots, N$ a.s. since λ_1 is the largest eigenvalue of Σ . Using $m(2) = 1$ and (3.38), we conclude that $x^\top O_j \Sigma O_j^\top x = \lambda_1 |x|^2$ for all $j = 1, \dots, N$ a.s. Since $x \in E_{\lambda_1}(\Sigma)$ was arbitrary, we infer that, a.s., the restriction of O_j to $E_{\lambda_1}(\Sigma)$ is an automorphism of $E_{\lambda_1}(\Sigma)$, $j = 1, \dots, N$. Using this fact, the above argument can be repeated consecutively for the remaining eigenvalues of Σ (in decreasing order) to conclude that, a.s., each O_j , $j = 1, \dots, N$ is an automorphism of $E_{\lambda_k}(\Sigma)$, $k = 2, \dots, n$. Since every vector $x \in \mathbb{R}^d$ can be written as a linear combination of normed eigenvectors of the eigenvalues $\lambda_1, \dots, \lambda_n$, we conclude that $\Sigma x = O_j \Sigma O_j^\top x$ for $j = 1, \dots, N$ a.s. and hence $\Sigma = O_j \Sigma O_j^\top$ for $j = 1, \dots, N$ a.s. Standard arguments then yield that $\Sigma = o\Sigma o^\top$ for all $o \in \mathbb{O}$. Using this, the martingale convergence theorem, (3.20) and the convergence $W_n \rightarrow W$ a.s., we conclude that

$$\begin{aligned} \Sigma &= \lim_{n \rightarrow \infty} \mathbb{E}[\Sigma | \mathcal{F}_n] = \lim_{n \rightarrow \infty} \sum_{|v|=n} L(v) \Sigma L(v)^\top \\ &= \lim_{n \rightarrow \infty} \sum_{|v|=n} \|L(v)\|^2 O(v) \Sigma O(v)^\top = W \Sigma \quad \text{a.s.} \end{aligned}$$

□

3.9 The proof of Proposition 1.1

In contrast to Eqs. (3.19) and (3.20), which allowed to determine ν and Σ , there is in general no such equation for W' due to additional contributions coming from the Lévy measure, see Eq. (3.18). Therefore, we have to postpone the identification of W' until the next section, and will prove Proposition 1.1 first. This is closely related to determining W' : Indeed, suppose that Z is a solution to (1.10), in particular, Z is \mathcal{F} -measurable. Iteration of (1.10) yields

$$Z = \sum_{|v|=n} L(v)[Z]_v \quad \text{a.s.} \quad (3.39)$$

for all $n \in \mathbb{N}_0$. Then consider the multiplicative martingales $(M_n(x))_{n \in \mathbb{N}_0}$ associated with Z (and its characteristic function ϕ). As in the proof of [5,

Proposition 4.17], we infer from the martingale convergence theorem,

$$\begin{aligned}
M(x) &= \lim_{n \rightarrow \infty} \prod_{|v|=n} \phi(L(v)^\top x) = \lim_{n \rightarrow \infty} \mathbb{E} \left[\exp \left(\sum_{|v|=n} i \langle L(v)^\top x, [Z]_v \rangle \right) \middle| \mathcal{F}_n \right] \\
&= \lim_{n \rightarrow \infty} \mathbb{E} \left[\exp \left(i \left\langle x, \sum_{|v|=n} L(v)[Z]_v \right\rangle \right) \middle| \mathcal{F}_n \right] = \lim_{n \rightarrow \infty} \mathbb{E}[\exp(i \langle x, Z \rangle) | \mathcal{F}_n] \\
&= \exp(i \langle x, Z \rangle) \quad \text{a.s.}
\end{aligned}$$

Hence, $Z = W'$ a.s. in (3.11) with Ψ denoting the characteristic exponent of M (in this case, Σ and ν vanish a.s.). Therefore, by Proposition 3.3, Z can be written in the form

$$Z = \lim_{n \rightarrow \infty} \sum_{|v|=n} \mathbb{E}[L(v)[Z]_v; |L(v)[Z]_v| \leq 1 | \mathcal{F}_n] \quad \text{a.s.} \quad (3.40)$$

Further, (3.25) gives

$$\mathbb{P}(|Z| > t) = o(t^{-\alpha}) \text{ as } t \rightarrow \infty. \quad (3.41)$$

This and the estimate (3.28) for L_β (see Eq. (3.26)) from Section 3.6 is everything we need to determine Z when $\alpha \neq 1$.

Proof (Proof of Proposition 1.1 in the case $\alpha \neq 1$) (a) Let $0 < \alpha < 1$. Then, using (3.40), (3.28), (1.5) and $W_n \rightarrow W$ a.s., we infer

$$\begin{aligned}
|Z| &= \lim_{n \rightarrow \infty} \left| \sum_{|v|=n} \mathbb{E}[L(v)[Z]_v; |L(v)[Z]_v| \leq 1 | \mathcal{F}_n] \right| \\
&\leq \limsup_{n \rightarrow \infty} \sum_{|v|=n} \|L(v)\| \mathbb{E}[| [Z]_v |; | [Z]_v | \leq \|L(v)\|^{-1} | \mathcal{F}_n] \\
&= \limsup_{n \rightarrow \infty} \sum_{|v|=n} \|L(v)\| L_1(\|L(v)\|^{-1}) = 0 \quad \text{a.s.}
\end{aligned}$$

(c) Suppose that $1 < \alpha < 2$, that Z satisfies (3.39) and that $\mathbb{P}(Z \neq 0) > 0$. Notice that (3.41) implies that $w := \mathbb{E}[Z]$ is finite, moreover, $Z \in \mathcal{L}^s$ for all $s < \alpha$. By standard martingale theory and (3.39), for any $1 \leq s < \alpha$,

$$\begin{aligned}
Z &= \lim_{n \rightarrow \infty} \mathbb{E}[Z | \mathcal{F}_n] = \lim_{n \rightarrow \infty} \mathbb{E} \left[\sum_{|v|=n} L(v)[Z]_v \middle| \mathcal{F}_n \right] \\
&= \lim_{n \rightarrow \infty} \sum_{|v|=n} L(v)w = \lim_{n \rightarrow \infty} Z_n w \quad \text{a.s. and in } \mathcal{L}^s.
\end{aligned}$$

Hence, $Z = Z^w$ and thus $w \neq 0$. Further, taking expectations in (3.39), we obtain

$$w = \mathbb{E}[Z] = \mathbb{E} \left[\sum_{j \geq 1} T_j[Z]_j \right] = \mathbb{E}[Z_1]w,$$

i.e., w is an eigenvector corresponding to the eigenvalue 1 of $\mathbb{E}[Z_1]$.

Conversely, suppose that $w = \mathbb{E}[Z]$ exists and is an eigenvector corresponding to the eigenvalue 1 of $\mathbb{E}[Z_1]$ and that $(Z_n w)_{n \in \mathbb{N}_0}$ is uniformly integrable. Then $Z_n w \rightarrow Z^w$ a.s. as $n \rightarrow \infty$ by the martingale convergence theorem and $\mathbb{E}[Z^w] = w$. Since w is an eigenvector, we have $w \neq 0$ and thus $\mathbb{P}(Z^w \neq 0) > 0$. One can then check that Z^w satisfies (3.39).

To finish the proof in the case $1 < \alpha < 2$, we have to show that $\mathbb{E}[|Z_1 w|^\beta] < \infty$ and $m(\beta) < 1$ is sufficient for the \mathcal{L}^β -boundedness of $(Z_n w)_{n \in \mathbb{N}_0}$. This, however, is a standard application of the Topchiř-Vatutin inequality for martingales. For details, we refer to [3, p. 182] and [71].

(d) Next, let $\alpha \geq 2$. If $w \in \mathbb{R}^d$ is an eigenvector corresponding to the eigenvalue 1 of Z_1 a.s., then $Z_n w = w$ a.s. for all $n \in \mathbb{N}_0$ and $Z^w = Z$ solves (3.39). For the converse implication, assume that Z solves (3.39) and that $\mathbb{P}(Z \neq 0) > 0$. By (3.41), $Z \in \mathcal{L}^s$ for all $s \in [1, \alpha]$. Pick some $s \in (1, 2)$ if $\alpha = 2$ and $s = 2$ if $\alpha > 2$ and use the lower bound in the Burkholder-Davis-Gundy inequality [31, Theorem 11.3.1] to derive lower bounds for $\mathbb{E}[|Z - w|^s]$ as in the proof of Theorem 2.3 in [43]. It follows that $\mathbb{E}[|Z - w|^s] = \infty$ unless $Z_1 w = w$ a.s. We refrain from providing more details. \square

The proof for the case $\alpha = 1$ is much more involved, some parts of it will be deferred to the Appendix. As a preparation, we show that it constitutes no loss of generality to assume the stronger assumption (A5) instead of (A4).

Lemma 3.6 *Suppose that (A1)–(A3) and either (A4) or (A4') hold. Let ϕ be a solution to (3.2). Then there is a weight sequence $\mathbf{T}' := (T'_j)_{j=1}^{N'}$ with $N' < \infty$ a.s. such that (A1)–(A4) resp. (A1)–(A4') persist to hold for \mathbf{T}' , ϕ is a solution to equation (3.2) associated with \mathbf{T}' as well and the martingale limit W defined in (1.8) is the same for \mathbf{T} and \mathbf{T}' . Moreover, $\|T'_j\| < 1$ a.s. for all $1 \leq j \leq N'$ and the group \mathbb{G} generated by \mathbf{T}' is the same as the one generated by \mathbf{T} . If \mathbf{T} satisfies (A4'), then also the group \mathbb{O} remains the same. If \mathbf{T} satisfies (A4), then (A5) holds for \mathbf{T}' .*

Proof We first deal with the case that (A4') holds. Set $\mathbf{T}' := (T(v))_{v \in \mathcal{C}(0)}$ (recall that $\|T(v)\| < 1$ a.s. for all $v \in \mathcal{C}(0)$). Then [44, Proposition 3.7] states that if $(T(v))_{|v|=1}$ satisfies the assumptions (A1)–(A4'), then so does the family $(T(v))_{v \in \mathcal{C}(0)}$. That W is the same for the sequences \mathbf{T} and \mathbf{T}' follows from an application of [55, Theorem 9]. The additional point which we have to take care of here is that we need to know that the closed multiplicative subgroup of $\mathbb{O}(d)$ generated by the $O(v)$, $v \in \mathcal{C}(0)$, which we denote by $\mathbb{O}^>$, equals \mathbb{O} . Clearly, $\mathbb{O}^> \subseteq \mathbb{O}$. Conversely, let $t > 0$ and $o \in \mathbb{O}(d)$ be such that to is in the support of T_j for some $j \in \mathbb{N}$. If we can show that $o \in \mathbb{O}^>$, then it follows that $\mathbb{O} \subseteq \mathbb{O}^>$. By (A2), there are $t' \in (0, 1)$ and $o' \in \mathbb{O}(d)$ such that $t'o'$ is in the support of some T_k for some $k \in \mathbb{N}$. Then $o' \in \mathbb{O}^>$ by the definition of $\mathcal{C}(0)$ and $\mathbb{O}^>$. For the minimal $m \in \mathbb{N}_0$ such that $t(t')^m < 1$, we have $o(o')^m \in \mathbb{O}^>$, which implies $o \in \mathbb{O}^>$.

Now we turn to the case that (A4) holds. Here, we obtain \mathbf{T}' in two successive steps. We start by considering as before the sequence $\tilde{\mathbf{T}} := (T(v))_{v \in \mathcal{C}(0)}$.

As above, it follows that (referring to the proof of [44, Proposition 3.7] for the moment assumptions in (A4)) that (A1)–(A3) as well as the spread-out assumption and the moment assumptions in (A4) carry over. Also, as above, W is the same for \mathbf{T} and $\tilde{\mathbf{T}}$. That $\eta := \mathbb{E}[\sum_{v \in \mathcal{C}(0)} \|T(v)\|^\alpha \delta_{T(v)}(\cdot)]$ is spread-out can be obtained by arguments similar to those given in the proof of [6, Lemma 1]. Hence there is an $n \in \mathbb{N}$ such that the n -fold convolution η^{*n} has a component which is continuous w.r.t. the Haar measure on $\mathbb{S}(d)$. Denote the corresponding density by f and assume without loss of generality that f has compact support and that $c \leq f \leq d$ for constants $0 < c < d$. Then η^{*2n} has a component with density f^{*2} , which is continuous [40, p. 295]. By the Fubini formula for the Haar measure on $\mathbb{S}(d) = \mathbb{R}_{>} \times \mathbb{O}(d)$ [33, Proposition 1.5.5],

$$\eta^{*2n} \geq \tilde{\gamma} H_{\mathbb{O}(d)|_D} \otimes H_{\mathbb{R}_{>|I}} \quad (3.42)$$

for some $\tilde{\gamma} > 0$ and open sets $D \subseteq \mathbb{O}(d)$, $I \subseteq \mathbb{R}_{>}$. Replacing $2n$ by $4n$ if necessary, we may assume that $D \subseteq \text{SO}(d)$. Since the latter group is compact and connected, we may invoke (the proof of) [16, Theorem 3] which gives that there is a $k \in \mathbb{N}$ such that $(H_{\mathbb{O}(d)|_D})^{*k} \geq \epsilon H_{\text{SO}(d)}$ for some $\epsilon > 0$. Then (3.42) yields that

$$\mu(A, B) := \eta^{*(2kn)}(A \times e^{-B}), \quad A \subseteq \mathbb{O}(d), B \subseteq \mathbb{R} \text{ measurable}$$

satisfies (M). On the other hand,

$$\mu = \mathbb{E} \left[\sum_{|v|=2kn} \|\tilde{L}(v)\|^\alpha \delta_{(\tilde{O}(v), -\log(\|\tilde{T}(v)\|))}(\cdot) \right],$$

where $(\tilde{L}(v))_{v \in \mathbb{V}}$ is the weighted branching process associated with $\tilde{\mathbf{T}}$. It is readily checked that assumptions (A1)–(A3) and the moment assumptions of (A4) persist to hold for $\mathbf{T}' := (\tilde{L}(v))_{|v|=2kn}$ and that also the martingale limit W is the same for \mathbf{T} and $\tilde{\mathbf{T}}$ and thus the assertion follows. \square

Remark 3.7 Under (A5), it holds that $\mathbb{U} = \mathbb{R}_{>} \times \text{SO}(d)$ or $\mathbb{U} = \mathbb{R}_{>} \times \mathbb{O}(d)$. Since the group \mathbb{U}' generated by \mathbf{T}' will always be a subgroup of the one generated by \mathbf{T} , it follows that under (A4), these are the only two possible cases for \mathbb{U} . Note that $\mathbb{U} = \mathbb{R}_{>} \times \mathbb{O}(d)$ while $\mathbb{U}' = \mathbb{R}_{>} \times \text{SO}(d)$ is possible (for example, if $\|T_j\| < 1$ and $\det(T_j) = -1$ for all $j = 1, \dots, N$). This is not a problem since both the (\mathbb{U}, α) - and (\mathbb{U}', α) -stable laws are just the d -dimensional rotational invariant α -stable laws.

Proof (Proof of Proposition 1.1 in the case $\alpha = 1$) By Lemma 3.6, we may assume throughout the proof, that $\|T_j\| \leq 1$ a.s. for all $1 \leq j \leq N$, and that (A5) holds if (A4) holds, i.e., we work with the sequence \mathbf{T}' instead of \mathbf{T} (but drop the superscript).

We begin with the converse implication of (b) and suppose that $w \in \mathbb{R}^d$ is an eigenvector corresponding to the eigenvalue 1 of $\mathbb{E}[Z_1]$. Then

$$|w| = |\mathbb{E}[Z_1]w| \leq \|\mathbb{E}[Z_1]\| \cdot |w| \leq \mathbb{E}[\|Z_1\|] \cdot |w| \leq \mathbb{E} \left[\sum_{j \geq 1} \|T_j\| \right] \cdot |w| = |w|.$$

Hence, equality must hold in this chain of inequalities and hence $T_j w = \|T_j\| w$ for all $j \in \mathbb{N}$ a.s. Consequently, $Z_n w = W_n w \rightarrow W w$ a.s. as $n \rightarrow \infty$ and $W w$ satisfies (3.39).

We are left with proving the direct implication in (b). Thus, assume that Z is \mathcal{F} -measurable and satisfies (3.39). First, we write $\mathbb{R}^d = V_+ + V_+^\perp$ where $V_+ = \{x \in \mathbb{R}^d : ox = x \text{ for all } o \in \mathbb{O}\}$. Both, V_+ and V_+^\perp are invariant subspaces of \mathbb{R}^d for every $o \in \mathbb{O}$. Write Z^{V_+} and $Z^{V_+^\perp}$ for the orthogonal projection of Z onto V_+ and V_+^\perp , respectively. We have

$$\begin{aligned} Z^{V_+} + Z^{V_+^\perp} &= Z = \sum_{|v|=n} L(v)[Z]_v = \sum_{|v|=n} L(v)[Z^{V_+}]_v + \sum_{|v|=n} L(v)[Z^{V_+^\perp}]_v \\ &= \sum_{|v|=n} \|L(v)\| [Z^{V_+}]_v + \sum_{|v|=n} L(v)[Z^{V_+^\perp}]_v. \end{aligned}$$

From linear independence (the left sum is in V_+ , the right sum is in V_+^\perp), we conclude that

$$\begin{aligned} Z^{V_+} &= \sum_{|v|=n} \|L(v)\| [Z^{V_+}]_v \quad \text{a.s. for all } n \in \mathbb{N}_0 \\ \text{and } Z^{V_+^\perp} &= \sum_{|v|=n} L(v)[Z^{V_+^\perp}]_v \quad \text{a.s. for all } n \in \mathbb{N}_0. \end{aligned}$$

The equation for Z^{V_+} can be reduced to one-dimensional equations and it follows from [5, Theorem 4.13] that $Z^{V_+} = W w$ for some $w \in V_+$. Taking expectations, we conclude that $w = \mathbb{E}[Z^{V_+}]$. Now notice that $V_+ = E_1(\mathbb{E}[Z_1])$ and thus w is an eigenvector of $\mathbb{E}[Z_1]$ or zero.

Observe that if $\text{SO}(d) \subseteq \mathbb{O}$, w has to be zero: Using the change of measure (3.4) with $f_k(L_1) := (O_1 w)_k$ for $1 \leq k \leq d$, we infer that

$$w = \mathbb{E} \left[\sum_{j \geq 1} \|T_j\| (O_j w) \right] = \mathbb{E}[O_1 w],$$

which implies $w = O_1 w$ a.s. Hence, $\text{SO}(d) \subseteq \mathbb{O}$ implies $w = 0$. Referring to Remark 3.7, we see that (A4) particularly implies $\text{SO}(d) \subseteq \mathbb{O}$.

It remains to show that $Z^{V_+^\perp} = 0$ a.s. We drop the V_+^\perp in the superscript and write Z for $Z^{V_+^\perp}$. We will use a variant of (3.40). To formulate it, consider the coming generation at time $t \geq 0$, which is $\mathcal{C}(t)$, defined by Eq. (3.6). The idea is to switch from genealogical generations in the branching process to particles living roughly at the same time (with $S(v)$ interpreted as the time of birth of particle v). The gain is that there is a better control over the birth times $S(v)$ when v ranges over $\mathcal{C}(t)$ than when it ranges of $\{|v| = n\}$. Instead of considering the multiplicative martingales $(M_n(x))_{n \in \mathbb{N}_0}$ for Z (defined via the characteristic function ϕ of Z), we consider

$$M_{\mathcal{C}(t)}(x) = \prod_{v \in \mathcal{C}(t)} \phi(L(v)^\top x), \quad x \in \mathbb{R}^d.$$

It can be checked along the lines of [2, Lemma 8.7(b)] and [5, Lemma 4.4] that $M_{\mathcal{C}(t)}(x) \rightarrow M(x)$ for all $x \in \mathbb{R}^d$ with $M(x) = \lim_{n \rightarrow \infty} M_n(x)$ a.s. Arguing as in the proof of [43, Lemma 3.6] gives

$$Z^h = \lim_{t \rightarrow \infty} \sum_{v \in \mathcal{C}(t)} \mathbb{E}[L(v)[Z]_v; |L(v)[Z]_v| \leq h \mid \mathcal{F}_{\mathcal{C}(t)}] \quad \text{a.s.} \quad (3.43)$$

for every $h > 0$ with $\nu(\{|x| = h\}) = 0$ a.s. Here, $\mathcal{F}_{\mathcal{C}(t)}$ is the σ -field that makes everything measurable in the weighted branching model defined by (\mathbf{C}, \mathbf{T}) that is born up to and including time t , see the proof of Lemma 8.7 in [2] for a rigorous definition. In the given situation, $\nu = 0$ a.s. and hence $Z = Z^h$ and $h > 0$ can be chosen arbitrarily, hence $h = 1$ is the most convenient choice. The key equation for us thus is

$$\begin{aligned} Z &= \lim_{t \rightarrow \infty} \sum_{v \in \mathcal{C}(t)} L(v) I(\|L(v)\|^{-1}) \\ &= \lim_{t \rightarrow \infty} \left(\sum_{v \in \mathcal{C}(t)} L(v) I(e^t) + \sum_{v \in \mathcal{C}(t)} L(v) (I(\|L(v)\|^{-1}) - I(e^t)) \right) \quad \text{a.s.} \end{aligned} \quad (3.44)$$

where $I(t) := \mathbb{E}[Z; |Z| \leq t]$, $t \geq 0$. For $0 \leq s < t$, we have

$$\begin{aligned} |I(t) - I(s)| &\leq \mathbb{E}[|Z|; s < |Z| \leq t] \\ &= \int_s^t \mathbb{P}(|Z| > x) \, dx - t\mathbb{P}(|Z| > t) + s\mathbb{P}(|Z| > s) \\ &\leq \int_s^t \mathbb{P}(|Z| > x) \, dx + s\mathbb{P}(|Z| > s). \end{aligned}$$

Using this for the last summand in the last line of (3.44), we infer

$$\begin{aligned} &\left| \sum_{v \in \mathcal{C}(t)} L(v) (I(\|L(v)\|^{-1}) - I(e^t)) \right| \\ &= \sum_{v \in \mathcal{C}(t)} \|L(v)\| \int_{e^t}^{\|L(v)\|^{-1}} \mathbb{P}(|Z| > x) \, dx + \sum_{v \in \mathcal{C}(t)} \|L(v)\| e^t \mathbb{P}(|Z| > e^t). \end{aligned} \quad (3.45)$$

The second sum tends to 0 a.s. as $t \rightarrow \infty$ by (3.41) and the fact that $\sum_{v \in \mathcal{C}(t)} \|L(v)\| = \mathbb{E}[W \mid \mathcal{F}_{\mathcal{C}(t)}] \rightarrow W$ a.s. Regarding the first sum, using (3.41) and $\|L(v)\| = \exp(-S(v))$, we obtain that, for arbitrary $\epsilon > 0$,

$$\sum_{v \in \mathcal{C}(t)} \|L(v)\| \int_{e^t}^{\|L(v)\|^{-1}} \mathbb{P}(|Z| > x) \, dx \leq \epsilon \sum_{v \in \mathcal{C}(t)} \|L(v)\| (S(v) - t)$$

for all sufficiently large t . By the law of large numbers for (single-type) general branching processes [66, Theorem 3.1], the last sum converges to a constant

multiple of W , see [5, p. 191] for a detailed argument. Since $\epsilon > 0$ was arbitrary, it remains to show that in (3.44)

$$\sum_{v \in \mathcal{C}(t)} L(v)I(e^t) \rightarrow 0 \quad \text{in probability as } t \rightarrow \infty. \quad (3.46)$$

Now notice that, for all $t \geq 1$, $I(e^t) \in V_+^\perp$ and, by (3.41),

$$|I(e^t)| \leq 1 + \int_1^{e^t} \mathbb{P}(|Z| > x) dx \leq \text{const} \cdot t.$$

Hence, (3.46) follows from Lemma 3.9. \square

Let us note one consequence out of the last step of the proof, which constitutes the analogue of [43, Lemma 4.9]:

Lemma 3.8 *Let X be a solution to (1.7) and let W^h be defined as in (3.13). For each $h > 0$, there is a finite constant $K > 0$ such that*

$$|W^h| \leq KW \quad \text{a.s.} \quad (3.47)$$

Proof Fix $h > 0$. It can be checked that the identity (3.43) holds with Z^h replaced by W^h . The subsequent estimates leading to Eq. (3.45) remain valid (also without restricting to V_+^\perp). Subsequently, one has to use the tail estimate (3.24) rather than (3.25) and hence obtains that

$$\left| W^h - \lim_{t \rightarrow \infty} \sum_{v \in \mathcal{C}(t)} L(v)I(e^t) \right| \leq K'W \quad \text{a.s.}$$

for some $K' \geq 0$. If now $\limsup_{t \rightarrow \infty} |I(e^t)| = \infty$, then $|W^h| = \infty$ on the set of survival, which yields a contradiction. Hence $\limsup_{t \rightarrow \infty} |I(e^t)| < \infty$ and the assertion follows.

Lemma 3.9 *Assume that (A1)–(A3), $\|T_j\| \leq 1$ a.s. for all $1 \leq j \leq N$ and that either (A4') or (A5) hold. Let $V_+ = \{x \in \mathbb{R}^d : ox = x \text{ for all } o \in \mathbb{O}\}$ be the space of \mathbb{O} -invariant vectors and let $V = V_+^\perp$. Further, denote by $(x_t)_{t \geq 0}$ a bounded sequence in V . Then*

$$\lim_{t \rightarrow \infty} t \left(\sum_{v \in \mathcal{C}(t)} L(v) \right) x_t = 0 \quad \text{in probability as } t \rightarrow \infty. \quad (3.48)$$

Proof Without loss of generality, we assume that $\sup_{t \geq 0} |x_t| \leq 1$.

Next notice that V_+ is an \mathbb{O} -invariant subspace of \mathbb{R}^d . Thus also its orthogonal complement V is an \mathbb{O} -invariant subspace of \mathbb{R}^d .

Step 1: First assume that (A4') holds. In particular, \mathbb{O} is finite. Let o_1, \dots, o_p denote the elements of \mathbb{O} with $o_1 = 1_{\mathbb{O}(d)}$. Then $((S(v), O(v))_{|v|=n})_{n \geq 0}$ defines

a multi-type branching random walk with type space \mathbb{O} . Pick an arbitrary $x \in V$. Then

$$\sum_{j=1}^p o_j x = 0. \quad (3.49)$$

Indeed, for $o \in \mathbb{O}$, $o\mathbb{O} = \mathbb{O}$ and, hence, $o(o_1x + \dots + o_px) = o_1x + \dots + o_px$. Consequently, $o_1x + \dots + o_px \in V_+$. On the other hand, $o_1x + \dots + o_px \in V$ since V is \mathbb{O} -invariant. This implies (3.49), which in turn implies

$$\frac{W}{p} \sum_{j=1}^p o_j x_t = 0$$

for every $t \geq 0$. Consequently,

$$\begin{aligned} t \left| \sum_{v \in \mathcal{C}(t)} L(v) x_t \right| &= t \left| \sum_{j=1}^p \left(\sum_{v \in \mathcal{C}(t): O(v)=o_j} e^{-S(v)} O(v) - \frac{W}{p} o_j \right) x_t \right| \\ &\leq \sum_{j=1}^p t \left| \left(\sum_{v \in \mathcal{C}(t): O(v)=o_j} e^{-S(v)} - \frac{W}{p} \right) o_j x_t \right| \\ &\leq \sum_{j=1}^p t \left| \sum_{v \in \mathcal{C}(t): O(v)=o_j} e^{-S(v)} - \frac{W}{p} \right|. \end{aligned}$$

The result now follows from [44, Theorem 2.14] once it has been checked that the assumptions of the cited theorem are satisfied in the present situation. The kind of checking that is needed to verify the assumptions can be found in the proof of [43, Lemma 3.14], it is here where the additional assumptions that $\|T_j\| \leq 1$ a.s., $1 \leq j \leq N$, enters.

Step 2: Next, we turn to the more delicate case where (A5) holds, hence \mathbb{O} is infinite. Suppose that $I_{t,1}, \dots, I_{t,p_t}$ is a partition of \mathbb{O} and, for $j = 1, \dots, p_t$, define $o_j := h_{t,j}^{-1} \int_{I_{t,j}} o H_{\mathbb{O}}(do)$ where $H_{\mathbb{O}}$ is the normed Haar measure on \mathbb{O} and $h_{t,j} = H_{\mathbb{O}}(I_{t,j})$. Due to the translation invariance of the Haar measure $H_{\mathbb{O}}$, the matrix $h_{t,1}o_1 + \dots + h_{t,p_t}o_{p_t}$ is invariant under multiplication with elements from \mathbb{O} and hence $(h_{t,1}o_1 + \dots + h_{t,p_t}o_{p_t})x = 0$ for all $x \in V$. Therefore,

$$\begin{aligned} t \left| \sum_{v \in \mathcal{C}(t)} L(v) x_t \right| &= t \left| \sum_{v \in \mathcal{C}(t)} L(v) x_t - W \sum_{j=1}^{p_t} h_{t,j} o_j x_t \right| \quad (3.50) \\ &\leq t \left| \sum_{j=1}^{p_t} \sum_{v \in \mathcal{C}(t): O(v) \in I_{t,j}} e^{-S(v)} (O(v) - o_j) x_t \right| \\ &\quad + t \left| \sum_{j=1}^{p_t} \left(\sum_{v \in \mathcal{C}(t): O(v) \in I_{t,j}} e^{-S(v)} - h_{t,j} W \right) o_j x_t \right|. \end{aligned}$$

Now assume that the partition is uniformly fine in the sense that

$$\max_{j=1, \dots, p_t} \sup_{o \in I_{t,j}} \|o - o_j\| = o(t^{-1}) \quad \text{as } t \rightarrow \infty. \quad (3.51)$$

Since $\mathbb{O} \subseteq \mathbb{O}(d)$ and $\mathbb{O}(d)$ is a $d(d-1)/2$ -dimensional smooth manifold, it has box-counting dimension $d(d-1)/2$, see e.g. [36, Section 3.2]. Hence, for $0 < \epsilon < \frac{\delta}{2} \wedge 1$, with δ as in (A4), we can choose the family of partitions in such a way that $p_t = o(t^{d(d-1)/2+\epsilon}) = o(t^{\ell-1+\epsilon})$ as $t \rightarrow \infty$, this will be assumed as well.

Then the summand in the second line of (3.50) vanishes as $t \rightarrow \infty$. We thus centre our attention on the summand in the third line. We will prove that

$$t \sum_{j=1}^{p_t} \left| \sum_{v \in \mathcal{C}(t): O(v) \in I_{t,j}} e^{-S(v)} - h_{t,j} W \right| \rightarrow 0 \quad \text{in probability as } t \rightarrow \infty. \quad (3.52)$$

We reformulate this problem in terms of a general branching process with type space \mathbb{O} (see [45] for definition and details). For a measurable set $I \subseteq \mathbb{O}$ define

$$\phi_I(t) = e^t \sum_{j \geq 1} e^{-S_j(\emptyset)} \mathbb{1}_{\{O_j(\emptyset) \in I\}} \mathbb{1}_{[0, S_j(\emptyset))}(t), \quad t \geq 0.$$

For $c > 0$, let $\phi_I^c(t) := \mathbb{1}_{[0, c]}(t) \phi_I(t)$ and notice that

$$[\phi_I^c]_v(t) = \mathbb{1}_{[0, c]}(t) e^t \sum_{j \geq 1} e^{-S_j(v)} \mathbb{1}_{\{O(v) O_j(v) \in I\}} \mathbb{1}_{[0, S_j(v))}(t), \quad t \geq 0.$$

The general branching process counted with characteristic ϕ_I^c is defined as

$$\mathcal{Z}^{\phi_I^c}(t) := \sum_{v \in \mathbb{V}} [\phi_I^c]_v(t - S(v)) = e^t \sum_{v \in \mathcal{C}(t): S(v|_{|v|-1}) \geq t-c} e^{-S(v)} \mathbb{1}_{\{O(v) \in I\}}, \quad t \geq 0.$$

We write $m_t^{\phi_I^c} := e^{-t} \mathbb{E}[\mathcal{Z}^{\phi_I^c}]$, which is finite since it is bounded by $\mathbb{E}[\mathcal{Z}^{\phi_0}] = 1$, see (3.7), and define

$$h_I^c := \frac{\mathbb{E}[S_1 \wedge c]}{\mathbb{E}[S_1]} H_{\mathbb{O}}(I). \quad (3.53)$$

For $0 < c < t$, we can then rewrite the left-hand side of (3.52) as follows:

$$\begin{aligned} t \sum_{j=1}^{p_t} \left| \sum_{v \in \mathcal{C}(t): O(v) \in I_{t,j}} e^{-S(v)} - h_{t,j} W \right| &= t \sum_{j=1}^{p_t} \left| e^{-t} \mathcal{Z}^{\phi_{I_{t,j}}}(t) - h_{t,j} W \right| \\ &\leq t \sum_{j=1}^{p_t} \left(e^{-t} \mathcal{Z}^{\phi_{I_{t,j}}}(t) - e^{-t} \mathcal{Z}^{\phi_{I_{t,j}}^c}(t) \right) + t \sum_{j=1}^{p_t} (h_{t,j} - h_{I_{t,j}}^c) W \\ &\quad + t \sum_{j=1}^{p_t} \left| e^{-t} \mathcal{Z}^{\phi_{I_{t,j}}^c}(t) - h_{I_{t,j}}^c W \right|. \end{aligned} \quad (3.54)$$

The first sum in (3.54) tends to 0 in \mathcal{L}^1 as $t \rightarrow \infty$ when $c = t/4$. Indeed, observe that $\mathbb{O} = \bigcup_{j=1}^{p_t} I_{t,j}$ implies that $\phi_{I_{t,1}}^c(t) + \dots + \phi_{I_{t,p_t}}^c(t) = \phi_{\mathbb{O}}^c(t)$, $t \geq 0$ and, hence,

$$\sum_{j=1}^{p_t} \mathcal{Z}^{\phi_{I_{t,j}}^c}(t) = \mathcal{Z}^{\phi_{\mathbb{O}}^c}(t), \quad t \geq 0. \quad (3.55)$$

Then, taking expectations in the first sum in (3.54) and using Eq. (3.7) as well as $\mathbb{E}[W] = 1$ leads to

$$\begin{aligned} & t \mathbb{E} \left[e^{-t} \sum_{j=1}^{p_t} \mathcal{Z}^{\phi_{I_{t,j}}}(t) - e^{-t} \sum_{j=1}^{p_t} \mathcal{Z}^{\phi_{I_{t,j}}^c}(t) \right] \\ &= t \mathbb{E} \left[\sum_{v \in \mathcal{C}(t)} e^{-S(v)} (1 - \mathbb{1}_{\{S(v|v|-1) \geq t-c\}}) \right] \\ &= t \mathbb{P}(S_{\tau(t)-1} < t-c) \leq t \mathbb{P}(S_{\tau(t)} - S_{\tau(t)-1} > c) \rightarrow 0 \text{ as } t \rightarrow \infty \end{aligned}$$

by Lemma C.1 if c grows linearly with t . From now on, we fix $c = t/4$. The expectation of the second sum in (3.54) is bounded by

$$t \frac{\mathbb{E}[S_1] - \mathbb{E}[S_1 \wedge c]}{\mathbb{E}[S_1]} = \frac{1}{\mathbb{E}[S_1]} t \int_{t/4}^{\infty} \mathbb{P}(S_1 > s) ds \rightarrow 0$$

as $t \rightarrow \infty$ since $\mathbb{E}[S_1^2] < \infty$ (which implies $P(S_1 > s) = o(s^{-2})$).

The remaining sum in (3.54) requires much more attention. We begin the proof by observing that $[\phi_{I_{t,j}}^c]_v(t - S(v)) = 0$ for all v with $S(v) \leq \frac{t}{2}$ due to the convention $c = t/4$. Consequently, for $j = 1, \dots, p_t$,

$$e^{-t} \mathcal{Z}^{\phi_{I_{t,j}}^c}(t) = \sum_{v \in \mathcal{C}(\frac{t}{2})} e^{-S(v)} e^{-(t-S(v))} [\mathcal{Z}^{\phi_{I_{t,j}}^c}]_v(t - S(v)).$$

Using this, we can estimate as follows:

$$\begin{aligned} & t \sum_{j=1}^{p_t} \left| e^{-t} \mathcal{Z}^{\phi_{I_{t,j}}^c}(t) - h_{I_{t,j}}^c W \right| \\ & \leq t \sum_{j=1}^{p_t} \sum_{v \in \mathcal{C}(\frac{t}{2})} e^{-S(v)} \left| e^{-(t-S(v))} [\mathcal{Z}^{\phi_{I_{t,j}}^c}]_v(t - S(v)) - m_{t-S(v)}^{\phi_{I_{t,j}}^c} \right| \\ & \quad + t \sum_{j=1}^{p_t} \sum_{v \in \mathcal{C}(\frac{t}{2}): S(v) \leq at} e^{-S(v)} \left| m_{t-S(v)}^{\phi_{I_{t,j}}^c} - h_{I_{t,j}}^c \right| \\ & \quad + t \sum_{j=1}^{p_t} \sum_{v \in \mathcal{C}(\frac{t}{2}): S(v) > at} e^{-S(v)} \left| m_{t-S(v)}^{\phi_{I_{t,j}}^c} - h_{I_{t,j}}^c \right| \\ & \quad + t \sum_{j=1}^{p_t} h_{I_{t,j}}^c \left| \sum_{v \in \mathcal{C}(\frac{t}{2})} e^{-S(v)} - W \right| =: \sum_{k=1}^4 J_k(t) \end{aligned}$$

for some (fixed) $a \in (\frac{1}{2}, 1)$. We consider each term separately. Using (3.53), the sum over $(h_{I_{t,j}}^c)_j$ in $J_4(t)$ is uniformly bounded by one, and [44, Proposition 4.3] gives that $\lim_{t \rightarrow \infty} |\sum_{v \in \mathcal{C}(\frac{t}{2})} e^{-S(v)} - W| = 0$ a.s. For $J_3(t)$, we change the order of the summation (the number of summands being finite a.s.) and use (3.55) and the change of measure (3.7) to obtain

$$\begin{aligned} \mathbb{E}[J_3(t)] &\leq t \mathbb{E} \left[\sum_{v \in \mathcal{C}(\frac{t}{2}): S(v) > at} e^{-S(v)} \sum_{j=1}^{p_t} (m_{t-S(v)}^{\phi_{I_{t,j}}^c} + h_{I_{t,j}}^c) \right] \\ &\leq 2t \mathbb{P}(S_{\tau(t/2)} - t/2 > (a - 1/2)t), \end{aligned}$$

which vanishes by [44, Lemma A.3]. Turning to $J_2(t)$, using that $p_t = o(t^{\ell-1+\epsilon})$ and $\mathbb{E}[\sum_{v \in \mathcal{C}(\frac{t}{2})} e^{-S(v)}] = 1$ (by (3.7)), we have that

$$\begin{aligned} \mathbb{E}[J_2(t)] &= t \mathbb{E} \left[\sum_{v \in \mathcal{C}(\frac{t}{2}): S(v) \leq at} e^{-S(v)} \sum_{j=1}^{p_t} |m_{t-S(v)}^{\phi_{I_{t,j}}^c} - h_{I_{t,j}}^c| \right] \\ &\leq t \sup_{s \geq (a-1/2)t} \sum_{j=1}^{p_t} |m_s^{\phi_{I_{t,j}}^c} - h_{I_{t,j}}^c| \\ &\leq \sup_{s \geq (a-1/2)t} \left(\frac{s}{a-1/2} \right)^{\ell+\epsilon} \max_{1 \leq j \leq p_t} |m_s^{\phi_{I_{t,j}}^c} - h_{I_{t,j}}^c|. \quad (3.56) \end{aligned}$$

To proceed further, we need a rate-of-convergence result from Markov renewal theory. Indeed, applying the change of measure,

$$\begin{aligned} m_t^{\phi_{I_{t,j}}^c} &= e^{-t} \mathbb{E} \left[\sum_{v \in \mathbb{V}} [\phi_{I_{t,j}}^c]_v(t - S(v)) \right] \\ &= e^{-t} \mathbb{E} \left[\sum_{v \in \mathbb{V}} e^{t-S(v)} \mathbb{1}_{[0,c]}(t - S(v)) \sum_{i \geq 1} e^{-S_i(v)} \mathbb{1}_{I_{t,j}}(O(vi)) \mathbb{1}_{[0,S_i(v)]}(t - S(v)) \right] \\ &= \sum_{n=0}^{\infty} \mathbb{E} \left[\sum_{|v|=n} e^{-S(v)} f_{t,j}^c(O(v), t - S(v)) \right] = \mathbb{E} \left[\sum_{n=0}^{\infty} f_{t,j}^c(O_n, t - S_n) \right], \end{aligned}$$

with

$$f_{t,j}^c(o, r) := \mathbb{1}_{[0,c]}(r) \mathbb{E}[\mathbb{1}_{I_{t,j}}(oO_1) \mathbb{1}_{[0,S_1]}(r)] \leq \mathbb{1}_{[0,\infty)}(r) \mathbb{P}(S_1 > r) =: g(r).$$

By (A4), the function g satisfies $g(r) = o(r^{\ell+\delta+1})$. From (3.53), we conclude

$$h_{I_{t,j}}^c = \frac{1}{\mathbb{E}[S_1]} \int_{\mathbb{R}} \int_{\mathbb{O}} f_{t,j}^c(o, r) H_{\mathbb{O}}(do) dr$$

and hence we can apply Proposition C.2 to deduce that the last term in (3.56) tends to zero as $t \rightarrow \infty$.

We finally consider $J_1(t)$ and proceed as in [44, pp. 735–736]. For fixed t , define

$$Z_{v,j} := e^{-(t-S(v))} [\mathcal{Z}_{I_{t,j}}^{\phi_{I_{t,j}}^c}]_v(t - S(v))$$

and similarly Z_v with $I_{t,j}$ replaced by \mathbb{O} , i.e., $Z_v = \sum_{j=1}^{p_t} Z_{v,j}$. Conditioned upon $\mathcal{F}_{\mathcal{C}(t/2)}$, the Z_v , $v \in \mathcal{C}(t/2)$ are independent. Let $Z'_{v,j} := Z_{v,j} \mathbb{1}_{\{Z_v \leq e^{S(v)}\}}$, $m'_{v,j} := \mathbb{E}[Z'_{v,j}]$, and $J'_1(t)$ as $J_1(t)$, but with $Z_{v,j}$ and $m_{t-S(v)}^{\phi_{I_{t,j}}^c}$ replaced by $Z'_{v,j}$ and $m'_{v,j}$, respectively. On the set $\{Z_v \leq e^{S(v)} \text{ for all } v \in \mathcal{C}(t/2)\}$,

$$\begin{aligned} J_1(t) &= J'_1(t) + t \sum_{j=1}^{p_t} \sum_{v \in \mathcal{C}(\frac{t}{2})} e^{-S(v)} (m_{t-S(v)}^{\phi_{I_{t,j}}^c} - m'_{v,j}) \\ &= J'_1(t) + t \sum_{v \in \mathcal{C}(\frac{t}{2})} e^{-S(v)} (m_{t-S(v)}^{\phi_{\mathbb{O}}^c} - m'_v). \end{aligned}$$

We want to prove that $J_1(t) \rightarrow 0$ in probability. To this end, we use the above decomposition and obtain for arbitrary $\eta > 0$,

$$\begin{aligned} \mathbb{P}(|J_1(t)| \geq \eta) &= \mathbb{E}[\mathbb{P}(|J_1(t)| \geq \eta | \mathcal{F}_{\mathcal{C}(\frac{t}{2})})] \\ &\leq \mathbb{E}\left[\sum_{v \in \mathcal{C}(\frac{t}{2})} \mathbb{P}(Z_v > e^{S(v)} | \mathcal{F}_{\mathcal{C}(\frac{t}{2})})\right] + \mathbb{E}[\mathbb{P}(|J'_1(t)| \geq \eta/2 | \mathcal{F}_{\mathcal{C}(\frac{t}{2})})] \\ &\quad + \frac{2t}{\eta} \mathbb{E}\left[\sum_{v \in \mathcal{C}(\frac{t}{2})} e^{-S(v)} (m_{t-S(v)}^{\phi_{\mathbb{O}}^c} - m'_v)\right]. \end{aligned}$$

The first and the last term can be dealt with as the corresponding terms in [44, pp. 735–736]. It remains to consider the middle term.

$$\begin{aligned} &\mathbb{E}[\mathbb{P}(|J'_1(t)| \geq \eta/2 | \mathcal{F}_{\mathcal{C}(\frac{t}{2})})] \\ &\leq \sum_{j=1}^{p_t} \mathbb{E}\left[\mathbb{P}\left(t \sum_{v \in \mathcal{C}(\frac{t}{2})} e^{-S(v)} |Z'_{v,j} - m'_{v,j}| \geq \eta/(2p_t) \mid \mathcal{F}_{\mathcal{C}(\frac{t}{2})}\right)\right] \\ &\leq \sum_{j=1}^{p_t} \frac{4p_t^2 t^2}{\eta^2} \mathbb{E}\left[\sum_{v \in \mathcal{C}(\frac{t}{2})} e^{-2S(v)} \text{Var}[Z'_{v,j} | \mathcal{F}_{\mathcal{C}(\frac{t}{2})}]\right] \\ &\leq \frac{4p_t^2 t^2}{\eta^2} \mathbb{E}\left[\sum_{v \in \mathcal{C}(\frac{t}{2})} e^{-2S(v)} \sum_{j=1}^{p_t} \mathbb{E}[(Z'_{v,j})^2 | \mathcal{F}_{\mathcal{C}(\frac{t}{2})}]\right] \\ &\leq \frac{4p_t^2 t^2}{\eta^2} \mathbb{E}\left[\sum_{v \in \mathcal{C}(\frac{t}{2})} e^{-2S(v)} \mathbb{E}[Z_v^2 \mathbb{1}_{\{Z_v \leq e^{S(v)}\}} | \mathcal{F}_{\mathcal{C}(\frac{t}{2})}]\right] \\ &= \frac{4p_t^2 t^2}{\eta^2} \mathbb{E}\left[\sum_{v \in \mathcal{C}(\frac{t}{2})} e^{-2S(v)} \mathbb{E}\left[h_{2\ell+\delta}(Z_v) \frac{Z_v^2}{h_{2\ell+\delta}(Z_v)} \mathbb{1}_{\{Z_v \leq e^{S(v)}\}} \mid \mathcal{F}_{\mathcal{C}(\frac{t}{2})}\right]\right] \\ &\leq \frac{4p_t^2 t^2}{\eta^2} \mathbb{E}\left[\sum_{v \in \mathcal{C}(\frac{t}{2})} e^{-S(v)} \frac{e^{S(v)}}{h_{2\ell+\delta}(e^{S(v)})}\right] \sup_{s \geq 0} \mathbb{E}\left[h_{2\ell+\delta}(e^{-s} \mathcal{Z}^{\phi_{\mathbb{O}}}(s))\right] \\ &\leq \frac{4}{\eta^2} \frac{e^{t/2} t^{2\ell+2\epsilon}}{h_{2\ell+\delta}(e^{t/2})} \sup_{s \geq 0} \mathbb{E}\left[h_{2\ell+\delta}(e^{-s} \mathcal{Z}^{\phi_{\mathbb{O}}}(s))\right] \rightarrow 0 \quad \text{as } t \rightarrow \infty. \end{aligned}$$

Here we used the triangular inequality, the independence of $Z'_{v,j}$ and $\mathcal{F}_{C(\frac{t}{2})}$, Chebyshev's inequality and the facts that $t \mapsto t^2/h_{2\ell+\delta}(t)$ and $t \mapsto t/h_{2\ell+\delta}(t)$ are increasing and decreasing, respectively, for large t , $t(\log t)^{2\ell+2\epsilon}/h_{2\ell+\delta}(t) \rightarrow 0$ as $t \rightarrow \infty$, and the finiteness of the supremum, which follows from (A4) and is proved as in [43, Lemma 3.14]. \square

3.10 Computing Ψ

In this section, we finish the determination of the \mathcal{F} -measurable Lévy triplet (W', Σ, ν) of solutions of the homogeneous equation. In order to do so, we will make use of some results proved in Section 4 below, which are postponed since their proofs do not require probabilistic tools like branching processes that are used in this section.

As a by-product, we prove Proposition 1.6. We remind the reader of the definition of the functions η_1^α , η_2^α , η^1 and the vector γ^1 in Eqs. (1.19)–(1.22). η_1^α , η_2^α are defined in terms of a (\mathbb{U}, α) -invariant Lévy measure ν^α , i.e., satisfying (3.29); η^1 and γ^1 are defined in terms of a measure ρ on \mathbb{S}^{d-1} . Note that (A4') implies $\mathbb{G} = \mathbb{R}$, hence $\mathbb{U} = \{t^Q : t > 0\} \times C_{\mathbb{U}}$, see Proposition 4.1. Here $t^Q := e^{(\ln t)Q}$, and we choose Q such that $\|t^Q\| = t$.

Proposition 3.10 *Assume (A1)–(A3), let ϕ be a solution to (3.2) and let $\Phi = \exp(\Psi)$ be the limit of the multiplicative martingales, given by Proposition 3.3.*

(a) *Let $\alpha \in (0, 1)$. There is a (\mathbb{U}, α) -invariant Lévy measure ν^α such that a.s.,*

$$\Psi(x) = -W|x|^\alpha \eta_1^\alpha(x) + iW|x|^\alpha \eta_2^\alpha(x) \quad \forall x \in \mathbb{R}^d. \quad (3.57)$$

(b) *Let $\alpha = 1$.*

(b1) *Assume (A4) in addition. There is a $c > 0$ such that a.s.,*

$$\Psi(x) = -Wc|x| \quad \forall x \in \mathbb{R}^d. \quad (3.58)$$

(b2) *Assume (A4') in addition. There is a $z \in \mathbb{R}^d$ with $\mathbb{E}[\sum_{j \geq 1} T_j]z = z$ and a finite $C_{\mathbb{U}}$ -invariant measure ρ on \mathbb{S}^{d-1} , satisfying $\int \langle x, s \rangle \rho(ds) = 0$ for all $x \in E_1(Q^\top)$, such that a.s.,*

$$\Psi(x) = iW\langle z, x \rangle + W(\eta^1(x) + i\langle \gamma^1, x \rangle) \quad \forall x \in \mathbb{R}^d. \quad (3.59)$$

(c) *Let $\alpha \in (1, 2)$. There is a (\mathbb{U}, α) -invariant Lévy measure ν^α such that a.s.,*

$$\Psi(x) = i\langle Z, x \rangle - W|x|^\alpha \eta_1^\alpha(x) + iW|x|^\alpha \eta_2^\alpha(x) \quad \forall x \in \mathbb{R}^d. \quad (3.60)$$

(d) *Let $\alpha = 2$. Then there is a positive semi-definite $d \times d$ matrix Σ satisfying $o^\top \Sigma o$ for all $o \in \mathbb{O}$ and a $z \in \mathbb{R}^d$ with $z = \sum_{j \geq 1} T_j z$ a.s., such that a.s.,*

$$\Psi(x) = i\langle z, x \rangle - W \frac{x^\top \Sigma x}{2} \quad \forall x \in \mathbb{R}^d. \quad (3.61)$$

(e) Let $\alpha > 2$. Then there is a $z \in \mathbb{R}^d$ with $z = \sum_{j \geq 1} T_j z$ a.s., such that a.s.,

$$\Psi(x) = i\langle z, x \rangle \quad \forall x \in \mathbb{R}^d. \quad (3.62)$$

Proof By Proposition 3.3, Ψ is a Lévy-Khintchine exponent (see (3.11)) with an \mathcal{F} -measurable Lévy triplet (W', Σ, ν) . By Lemma 3.4, $\nu = W\bar{\nu}$ a.s. for a deterministic (\mathbb{U}, α) -invariant Lévy measure $\bar{\nu}$. By Lemma 3.5, $\Sigma = W\Sigma$ a.s. for a deterministic covariance matrix Σ satisfying $\Sigma = o\Sigma o^\top$ for all $o \in \mathbb{O}$. Moreover, $\bar{\nu} = 0$ if $\alpha \geq 2$ and $\Sigma = 0$ unless $\alpha = 2$. The burden of the proof is to determine the random shift W' . We now consider the cases separately.

(a) Let $0 < \alpha < 1$. Using the evaluation of the Lévy integral in Lemma B.1, we obtain from (3.11) that for all $x \in \mathbb{R}^d \setminus \{0\}$,

$$\begin{aligned} \Psi(x) &= i\langle W', x \rangle + W(-|x|^\alpha \eta_1^\alpha(x) + i|x|^\alpha \eta_2^\alpha(x) + i\langle \gamma^\alpha, x \rangle) \\ &= i\langle W' + \gamma^\alpha W, x \rangle - W|x|^\alpha \eta_1^\alpha(x) + iW|x|^\alpha \eta_2^\alpha(x), \end{aligned} \quad (3.63)$$

for functions $\eta_1^\alpha, \eta_2^\alpha$ defined as in (1.19) and (1.20), resp., in terms of $\nu^\alpha := \bar{\nu}$. Using that $\eta_j^\alpha(u^\top x) = \eta_j^\alpha(x)$, $j = 1, 2$, and (3.17), we infer that for all $n \in \mathbb{N}$

$$\begin{aligned} \Psi(x) &= \sum_{|v|=n} [\Psi]_v (L(v)^\top x) \\ &= i \sum_{|v|=n} \langle L(v)[W' + \gamma^\alpha W]_v, x \rangle - \sum_{|v|=n} \|L(v)\|^\alpha [W]_v |x|^\alpha \eta_1^\alpha(x) \\ &\quad + i \sum_{|v|=n} \|L(v)\|^\alpha [W]_v |x|^\alpha \eta_2^\alpha(x) \quad \text{a.s.} \end{aligned} \quad (3.64)$$

Combining (3.63) and (3.64) and linear independence of 1 and i , we obtain

$$\begin{aligned} &\langle W' + \gamma^\alpha W, x \rangle + W|x|^\alpha \eta_2^\alpha(x) \\ &= \sum_{|v|=n} \langle L(v)[W' + \gamma^\alpha W]_v, x \rangle + \sum_{|v|=n} [W]_v \|L(v)\|^\alpha |x|^\alpha \eta_2^\alpha(x). \end{aligned} \quad (3.65)$$

Dividing by $|x|$ and letting $|x| \rightarrow \infty$, we obtain

$$\langle W' + \gamma^\alpha W, y \rangle = \sum_{|v|=n} \langle L(v)[W' + \gamma^\alpha W]_v, y \rangle \quad (3.66)$$

for all $y \in \mathbb{S}^{d-1}$, hence $W' + \gamma^\alpha W$ is a solution to (1.10), i.e., is an endogenous fixed point. Since $\alpha < 1$, Proposition 1.1 yields that $W' + \gamma^\alpha W = 0$ a.s. Then (3.57) follows from (3.63).

(c) For $1 < \alpha < 2$, we start from Eq. (3.65). Dividing by $|x|$, but considering $|x| \rightarrow 0$ this time, we obtain the identity (3.66). Hence, $W' + \gamma^\alpha W$ is an endogenous fixed point. Proposition 1.1 thus implies (3.60).

(b1) Assumption (A4) implies $\mathbb{U} = \mathbb{R}_> \times \mathbb{O}$ with $\mathbb{O} = \text{SO}(d)$ or $\mathbb{O} = \mathbb{O}(d)$, see Remark 3.7. Thus, the r.v. with Lévy triplet $(0, 0, \bar{\nu})$ is rotation invariant

and 1-stable. By [72, Theorem 14.14], its characteristic exponent equals $-c|x|$ for some $c > 0$ and we obtain that a.s.

$$\Psi(x) = i\langle W', x \rangle - Wc|x|.$$

Using (3.17) and considering real and imaginary part separately, we obtain that

$$W' = \sum_{|v|=n} L(v)[W']_v,$$

and Proposition 1.1 yields that $W' = 0$ a.s. due to the assumption (A4).

(b2) Let ρ be the spherical component of the Lévy measure $\bar{\nu}$, given by Proposition 4.3. We start by proving that $\int \langle x, s \rangle \rho(ds) = 0$ for all $x \in E_1(Q^\top)$. We consider two different cases:

1. Suppose $x \in E_1(Q^\top) \cap E_1(C_{\mathbb{U}}) =: V'$, i.e., $u^\top x = \|u\| x$ for all $u \in \mathbb{U}$. For this case, we adjust the proof given of [43, Theorem 4.10 (b1)] to the present situation. Note that on V' , (1.7) reduces to a fixed-point equation of a smoothing transformation with nonnegative scalar weights. Let e_1, \dots, e_k denote an orthonormal basis of V' and write $\tilde{W}_j := \langle e_j, W' + \gamma W \rangle$ where γ is as in (B.3). Further, let $s_j := \langle e_j, s \rangle$ for $j = 1, \dots, k$ and $s \in \mathbb{R}^d$. Thus, using (B.3) and the linear independence of i and 1 , we obtain for all $r \in \mathbb{R}_{>}$, $j = 1, \dots, k$, a.s.

$$\begin{aligned} r\tilde{W}_j - W \frac{2}{\pi} \int_{\mathbb{S}^{d-1}} rs_j \log(|rs_j|) \rho(ds) \\ = r \sum_{v=n} \|L(v)\| [\tilde{W}_j]_v - \sum_{v=n} \|L(v)\| \log(\|L(v)\|) [W]_v \frac{2}{\pi} \int_{\mathbb{S}^{d-1}} rs_j \rho(ds) \\ - \sum_{v=n} \|L(v)\| [W]_v \frac{2}{\pi} \int_{\mathbb{S}^{d-1}} rs_j \log(|rs_j|) \rho(ds). \end{aligned} \quad (3.67)$$

Assuming for a contradiction that $\int_{\mathbb{S}^{d-1}} s_j \rho(ds) \neq 0$ for some j , we choose $r > 0$ such that $\int_{\mathbb{S}^{d-1}} rs_j \log |rs_j| \rho(ds)$ vanishes. Hence, upon dividing by r ,

$$\tilde{W}_j = \sum_{v=n} \|L(v)\| [\tilde{W}_j]_v - \sum_{v=n} \|L(v)\| \log(\|L(v)\|) [W]_v \frac{2}{\pi} \int_{\mathbb{S}^{d-1}} s_j \rho(ds). \quad (3.68)$$

By Lemma 3.8, Eqs. (3.12) and (3.13) and Lemma 3.4, we deduce that there is some $K \geq 0$ such that $|\tilde{W}_j| \leq KW$ a.s., and consequently also $|\tilde{W}_j|_v \leq K[W]_v$ a.s. for all $v \in \mathbb{V}$. This together with (3.68) yields that a.s.

$$\left| \sum_{v=n} \|L(v)\| \log(\|L(v)\|) [W]_v \frac{2}{\pi} \int_{\mathbb{S}^{d-1}} s_j \rho(ds) \right| \leq 2KW.$$

The assumption $\int_{\mathbb{S}^{d-1}} s_j \rho(ds) \neq 0$ implies that the left-hand side tends to ∞ a.s. since $\lim_{n \rightarrow \infty} \sup_{|v|=n} \|L(v)\| = 0$ a.s. by (1.5) and $\sum_{|v|=n} \|L(v)\| [W]_v = W$ a.s. Contradiction! Hence, $\int_{\mathbb{S}^{d-1}} \langle x, s \rangle \rho(ds) = 0$ for all $x \in V'$.

2. Suppose $x \in E_1(C_{\mathbb{U}})^\perp$. Since ρ is $C_{\mathbb{U}}$ -invariant (see Proposition 4.3), it follows that $s_0 := \int s \rho(ds)$ is $C_{\mathbb{U}}$ -invariant as well, thus $0 = \langle x, s_0 \rangle = \int \langle x, s \rangle \rho(ds)$.

Hence, by 1. and 2. together,

$$\int \langle x, s \rangle \rho(ds) = 0 \quad \text{for all } x \in E_1(Q^\top). \quad (3.69)$$

By Corollary 4.2 below, the infinitely divisible law with Lévy triplet $(0, 0, \bar{\nu})$ is operator-stable with exponent Q . For such laws, [59, Theorem 13] shows that validity of (3.69) is equivalent to the existence of γ^1 (given by the fomula (1.22), cf. [59, Proposition 12 and Eq. (19)]) such that

$$\tilde{\Psi}(x) := i\langle \gamma^1, x \rangle + \eta^1(x)$$

satisfies $\tilde{\Psi}((t^Q)^\top x) = t\tilde{\Psi}(x)$ for all $t > 0$. Moreover, since t^Q commutes with $C_{\mathbb{U}}$ (see Prop. 4.1), the $C_{\mathbb{U}}$ -invariance of ρ and the definition of γ^1 imply that γ^1 is $C_{\mathbb{U}}$ -invariant, and so is η^1 . Hence, $\tilde{\Psi}((t^Q)^\top o^\top x) = t\tilde{\Psi}(x)$ for all $t > 0$, $o \in C_{\mathbb{U}}$, i.e., $\tilde{\Psi}$ is the exponent of a strictly $(\mathbb{U}, 1)$ -stable law.

Thus we can write

$$\Psi(x) = i\langle W' - \gamma^1 W, x \rangle + W\tilde{\Psi}(x), \quad (3.70)$$

and this equals, using (3.17) and the strict $(\mathbb{U}, 1)$ -stability of $\tilde{\Psi}$,

$$\begin{aligned} \Psi(x) &= i\langle \sum_{|v|=n} L(v)[W' - \gamma^1 W]_v, x \rangle + \sum_{|v|=n} \tilde{\Psi}(x) \|L(v)\| [W]_v \\ &= i\langle \sum_{|v|=n} L(v)[W' - \gamma^1 W]_v, x \rangle + W\tilde{\Psi}(x). \end{aligned} \quad (3.71)$$

Subtracting (3.71) from (3.70), we infer that $W' - \gamma^1 W$ is an endogenous fixed point, hence equals zW for some z with $\mathbb{E}[\sum_{j \geq 1} T_j]z = z$ by Proposition 1.1. All in all,

$$\Psi(x) = zW + \tilde{\Psi}(x).$$

(d) $\alpha = 2$ implies $\nu = 0$ a.s. and hence

$$\Psi(x) = i\langle W', x \rangle - W \frac{x^\top \Sigma x}{2}.$$

The claimed properties of Σ are proved in Lemma 3.5. Using again the linear independence of 1 and i and (3.17), we deduce, since $x \in \mathbb{R}^d \setminus \{0\}$ is arbitrary, that

$$W' = \sum_{|v|=n} L(v)[W']_v,$$

hence W' satisfies (1.10). By Proposition 1.1, either $W' = 0$ or $W' = w$ for a deterministic $w \neq 0$ which satisfies $w = \sum_{j \geq 1} T_j w$ a.s.

(e) In this case, ν and Σ vanish, W' can be identified as in (d). \square

Proof (Proof of Proposition 1.6) The notion of (U, α) -stability implies, using uniqueness of the Lévy triple $(\gamma, \Sigma, \nu^\alpha)$, that ν^α satisfies (3.29) and that $o^\top \Sigma o$ for all $o \in O$. For $\alpha \neq 1$, we can argue as before, using the identity $\Psi(u^\top x) = \|u\|^\alpha \Psi(x)$ and letting $\|u\| \rightarrow \infty$ resp. $\|u\| \rightarrow 0$ to prove that only $\eta_{1,2}^\alpha$ or Σ remain.

If $\alpha = 1$ and $U = \{t^Q : t > 0\} \times C$, then a $(U, 1)$ -stable law is in particular operator-stable with exponent Q (see Section 4.2), and necessarily of the form $(\gamma, 0, \nu^\alpha)$, where $\int f(x) \nu^\alpha(dx) = \int_{\mathbb{R}_>} \int_{\mathbb{S}^{d-1}} f(t^Q s) \frac{1}{t^Q} \rho(ds) dt$. Then [59, Proposition 12 and Theorem 13] give that (3.69) is equivalent to the existence of $\gamma \in \mathbb{R}^d$ such that $(\gamma, 0, \nu^\alpha)$ is strictly operator-stable. In addition, if (3.69) holds, then $(\gamma^1, 0, \nu^\alpha)$ is strictly operator-stable, and γ^1 inherits C -invariance from ρ . Thus, $(\gamma^1, 0, \nu^\alpha)$ is strictly $(U, 1)$ -stable and if $(\gamma, 0, \nu^\alpha)$ is strictly $(U, 1)$ -stable as well, then also $(\gamma^1 - \gamma, 0, 0)$ is $(U, 1)$ -stable, which implies that $z := \gamma^1 - \gamma$ satisfies $u^\top z = \|u\| z$ for all $u \in U$. \square

Propositions 1.6 and 3.10 together show that all solutions of the homogeneous equation are of the form $Z + Y_W$. It remains to solve the inhomogeneous equation.

3.11 Proof of Theorem 1.5: The converse inclusion

Since we have determined all solutions to the homogeneous equation in Proposition 3.10 above, we can now finish the proof of our main result by proving the converse inclusion in Theorem 1.5 (the direct inclusion has already been proved in Section 3.3).

Proof (Proof of Theorem 1.5: The converse inclusion) Let X be a solution to (1.1) with characteristic function ϕ . Write $\Phi_n(x) := \prod_{|v|=n} \phi(L(v)^\top x)$, and notice that the multiplicative martingale associated with ϕ takes the form $M_n(x) = \exp(i\langle x, W_n^* \rangle) \cdot \Phi_n(x)$ (see (3.8)). The assumption that $W_n^* \rightarrow W^*$ in probability implies

$$\Phi_n(x) \rightarrow M(x) / \exp(i\langle W^*, x \rangle) =: \Phi(x)$$

in probability. Arguing as in the proof of [4, Theorem 4.2], it follows that $\psi(x) := \mathbb{E}[\Phi(x)]$ satisfies the functional equation (3.2) of the homogeneous smoothing transform and that $\Phi(x)$ equals the limit of the multiplicative martingale associated with $\psi(x)$, hence $\Phi(x) = \exp(\Psi(x))$ with $\Psi(x)$ given by Proposition 3.10. We conclude that

$$\phi(x) = \mathbb{E}[M(x)] = \mathbb{E}[\exp(i\langle W^*, x \rangle) \Phi(x)] = \mathbb{E}[\exp(i\langle W^*, x \rangle + \Psi(x))].$$

\square

4 Matrices and measures invariant under actions of similarity groups

In this section, we study the property of (U, α) -stability in detail, for arbitrary closed subgroups $U \subseteq \mathbb{S}(d)$. We start by describing the general structure of such groups. This will allow us to relate (U, α) -stable laws to operator semi-stable laws, and to characterize Lévy measures and covariance matrices, satisfying the invariance properties (3.29) and (3.37), respectively.

4.1 Structure of U and polar coordinates

Let $U \not\subseteq \mathbb{O}(d)$ be a closed subgroup of the similarity group $\mathbb{S}(d)$, a particular case of which is \mathbb{U} , the closed subgroup generated by the T_j , $j = 1, \dots, N$. We write G for the image of U under the group homomorphism $u \mapsto \|u\|$ and distinguish between the discrete case $G = r^{\mathbb{Z}}$ for some $0 < r < 1$, and the continuous case $G = \mathbb{R}_{>}$. As before, $t^Q := e^{(\ln t)Q}$, and the right-hand side denotes the matrix exponential.

Proposition 4.1 *Let $C_U := U \cap \mathbb{O}(d)$. Then there is a subgroup $A_U \subseteq U$ isomorphic to G such that*

$$U \simeq A_U \ltimes C_U,$$

in particular, every $u \in U$ has a unique representation $u = ac$ with $a \in A_U$, $c \in C_U$. Moreover, in the

- *discrete case:* $A_U = \{A^n : n \in \mathbb{Z}\}$ for some $A \in U$ with $\|A\| = r$,
- *continuous case:* $A_U = \{t^Q : t \in \mathbb{R}_{>}\}$ for a $d \times d$ -matrix $Q = Q' + cI_d$, where Q' is skew symmetric and $c \neq 0$. Q can be chosen in such a way that A_U and C_U commute.

Proof The structure of U is given by Proposition C.1 in [24], commutativity in the continuous case is proved in Proposition D.13 in [24]. It is proved there that (in the continuous case) Q is an element of the Lie Algebra of U , i.e., $e^Q \in U \subseteq \mathbb{S}(d)$. Any orthogonal matrix is the exponential of a skew symmetric matrix, hence a similarity matrix u with $\|u\| \neq 1$ is the product of an orthogonal matrix times a scalar multiple of the identity matrix, which we represent by $e^{(\ln c)I_d}$. \square

The multiplication $\cdot : (A_U \ltimes C_U)^2 \rightarrow A_U \ltimes C_U$ is defined via the conjugation action induced on C_U by elements of A_U :

$$(a_1, c_1) \cdot (a_2, c_2) = (a_1 a_2, a_2^{-1} c_1 a_2 c_2), \quad (a_1, c_1), (a_2, c_2) \in A_U \times C_U.$$

Notice that $a_2^{-1} c_1 a_2 c_2 \in C_U$ since C_U is a normal subgroup of U . Further note that when all elements of A_U commute with all elements of C_U , multiplication on $A_U \ltimes C_U$ simplifies to

$$(a_1, c_1) \cdot (a_2, c_2) = (a_1 a_2, c_1 c_2), \quad (a_1, c_1), (a_2, c_2) \in A_U \times C_U$$

and hence, in this case, $A_U \ltimes C_U = A_U \times C_U$, that is, U is isomorphic to the direct product $A_U \times C_U$.

According to the two cases considered, we introduce generalized polar coordinates as follows. Let

$$S_U := \begin{cases} \{x \in \mathbb{R}^d : |x| = 1\} = \mathbb{S}^{d-1} & \text{if } G = \mathbb{R}_{>}, \\ \{x \in \mathbb{R}^d : r \leq |x| < 1\} & \text{if } G = r^{\mathbb{Z}}, 0 < r < 1. \end{cases} \quad (4.1)$$

Then any $x \in \mathbb{R}^d \setminus \{0\}$ has a unique representation $x = as$ with $s \in S_U$ and $a \in A_U$. Notice that in general, a is not a scalar. For example, in the setting of cyclic Pólya urns, Section 2.1.2,

$$A_U = \{t^\zeta : t \in \mathbb{R}_{>}\}, \quad C_U = \{\zeta^k : 0 \leq k < b\}, \quad S_U = \mathbb{S}^1,$$

where $\zeta = \cos(2\pi/b) + i\sin(2\pi/b)$ is a primitive b th root of unity.

4.2 Operator (semi)stable laws

An infinitely divisible law on \mathbb{R}^d with characteristic exponent Ψ is called (A, c) -operator semistable if there is a $d \times d$ -matrix A , $b \in \mathbb{R}^d$ and $c \in (0, 1)$ such that

$$\Psi(A^\top x) = c\Psi(x) + i\langle x, b \rangle \quad \text{for all } x \in \mathbb{R}^d, \quad (4.2)$$

see [39, Definition 1.3.6]. It is called *operator stable* with exponent Q if there is a matrix Q and a mapping $s \mapsto b(s) \in \mathbb{R}^d$ such that

$$\Psi((t^Q)^\top x) = t\Psi(x) + i\langle x, b(t) \rangle \quad \text{for all } x \in \mathbb{R}^d, t > 0, \quad (4.3)$$

see [39, Definition 1.3.11]. The law is called *strictly operator (semi)stable* if $b = 0$ or $b(t) \equiv 0$, respectively. Recalling the definition of (U, α) -stability in (1.12), and using the structure of U given in Proposition 4.1, we obtain the following Corollary.

Corollary 4.2 *Let η be a (strictly) (U, α) -stable law.*

1. *If $A_U = \{A^n : n \in \mathbb{Z}\}$ with $\|A\| < 1$, then (4.2) holds with $c = \|A\|^\alpha$, i.e., η is $(A, \|A\|^\alpha)$ (strictly) operator semistable.*
2. *If $A_U = \{e^{sQ} : s \in \mathbb{R}\}$, then (4.3) holds upon rescaling Q such that $\|t^Q\|^\alpha = t$, i.e., η is (strictly) operator stable with exponent Q .*

Notice that C_U did not play a role in the above considerations, therefore (U, α) -stability is more restrictive than operator (semi)stability.

4.3 Lévy measures invariant under similarity transformations

Using the generalized polar coordinates introduced in Section 4.1 above, we can now describe the structure of Lévy measures satisfying (3.29). Of course, here \mathbb{U} can be any closed subgroup of $\mathbb{S}(d)$. We write $H_{A_{\mathbb{U}}}$ for the Haar measures on $A_{\mathbb{U}} \simeq \mathbb{G}$, which is the counting measure in the discrete case, and the image of dt/t under the map $t \mapsto t^Q$ in the continuous case.

Proposition 4.3 *Let $\bar{\nu}$ be a Lévy measure on $\mathbb{R}^d \setminus \{0\}$. Then the following assertions are equivalent:*

- (i) $\bar{\nu}$ satisfies (3.29).
- (ii) There is a $C_{\mathbb{U}}$ -invariant finite measure ρ on $S_{\mathbb{U}}$ such that for all $\bar{\nu}$ -integrable $f : \mathbb{R}^d \setminus \{0\} \rightarrow \mathbb{R}$,

$$\int f(x) \bar{\nu}(dx) = \int_{A_{\mathbb{U}}} \int_{S_{\mathbb{U}}} f(ax) \|a\|^{-\alpha} \rho(dx) H_{A_{\mathbb{U}}}(da). \quad (4.4)$$

Notice that since $A_{\mathbb{U}}$ is not compact, the Haar measure $H_{A_{\mathbb{U}}}$ on $A_{\mathbb{U}}$ is unique up to a positive scaling constant only. In the discrete case, we stipulate that $H_{A_{\mathbb{U}}}$ is the counting measure. In the continuous case, we stipulate that $H_{A_{\mathbb{U}}}$ is such that the pushforward measure of $H_{A_{\mathbb{U}}}$ under the map $A_{\mathbb{U}} \rightarrow \mathbb{R}_{>}$, $a \mapsto \|a\|$ is dt/t .

Proof We consider the discrete and continuous case separately.

In the discrete case, (3.29) yields that $\bar{\nu}(A^{-1}\cdot) = \|A\|^{\alpha} \bar{\nu}$. Recall the definition of $S_{\mathbb{U}}$ from (4.1). Setting $\rho := \bar{\nu}(\cdot \cap S_{\mathbb{U}})$, we first observe that by (3.29), $\bar{\nu}$ is $C_{\mathbb{U}}$ -invariant and hence so is ρ . Further, [39, Theorem 1.4.5] implies that for any Borel set $B \subseteq \mathbb{R}^d \setminus \{0\}$,

$$\begin{aligned} \bar{\nu}(B) &= \sum_{n=-\infty}^{\infty} \|A\|^{n\alpha} \rho((A^n B) \cap S_{\mathbb{U}}) = \int_{A_{\mathbb{U}}} \|a\|^{-\alpha} \rho((a^{-1}B) \cap S_{\mathbb{U}}) H_{A_{\mathbb{U}}}(da) \\ &= \int_{A_{\mathbb{U}}} \|a\|^{-\alpha} \int_{S_{\mathbb{U}}} \mathbb{1}_B(ax) \rho(dx) H_{A_{\mathbb{U}}}(da). \end{aligned}$$

This proves (i) \Rightarrow (ii) in the discrete case.

For the converse implication, we invoke [39, Theorem 1.4.4] which gives that each measure satisfying (4.4) is a Lévy measure with $\bar{\nu}(A^{-n}\cdot) = \|A\|^{n\alpha} \bar{\nu}$, i.e., $\bar{\nu}(a^{-1}\cdot) = \|a\|^{\alpha} \bar{\nu}$ for all $a \in A_{\mathbb{U}}$. Let $\mathbb{U} \ni u = a_0 c$ with $a_0 \in A_{\mathbb{U}}$, $c \in C_{\mathbb{U}}$ and let B be a Borel subset of $\mathbb{R}^d \setminus \{0\}$. In general, $A_{\mathbb{U}}$ and $C_{\mathbb{U}}$ do not commute, but since $(a_0 c)^{-1}$ is an element of \mathbb{U} with norm $\|(a_0 c)^{-1}\| = \|a_0\|^{-1}$, there is

a $c' \in C_{\mathbb{U}}$ such that $(a_0 c)^{-1} = a_0^{-1} c'^{-1}$. Then

$$\begin{aligned}
 \bar{\nu}(c^{-1} a_0^{-1} B) &= \bar{\nu}(a_0^{-1} c'^{-1} B) = \|a_0\|^\alpha \bar{\nu}(c'^{-1} B) \\
 &= \|a_0\|^\alpha \int_{A_{\mathbb{U}}} \int_{S_{\mathbb{U}}} \mathbb{1}_B(c' a x) \|a\|^{-\alpha} \rho(dx) H_{A_{\mathbb{U}}}(da) \\
 &= \|a_0\|^\alpha \int_{A_{\mathbb{U}}} \int_{S_{\mathbb{U}}} \mathbb{1}_B(a c'_a x) \|a\|^{-\alpha} \rho(dx) H_{A_{\mathbb{U}}}(da) \\
 &= \|a_0\|^\alpha \int_{A_{\mathbb{U}}} \int_{S_{\mathbb{U}}} \mathbb{1}_B(ax) \|a\|^{-\alpha} \rho(dx) H_{A_{\mathbb{U}}}(da) \\
 &= \|a_0 c\|^\alpha \bar{\nu}(B),
 \end{aligned}$$

where the $c'_a = a^{-1} c' a \in C_{\mathbb{U}}$. In the next-to-last line, the $C_{\mathbb{U}}$ -invariance of ρ was used. Thus (ii) \Rightarrow (i) is proved in the discrete case.

Turning to the implication (i) \Rightarrow (ii) in the continuous case, choose Q in such a way that $\|t^Q\|^\alpha = t$. Then (3.29) for $(t^Q)^{-1}$ becomes $\bar{\nu}((t^Q)^{-1} \cdot) = t \bar{\nu}(\cdot)$. Define

$$\rho(B) := \bar{\nu}(\{t^Q x : x \in B, t \geq 1\}), \quad B \subseteq \mathbb{S}^{d-1}. \quad (4.5)$$

By Proposition 4.1, t^Q and $C_{\mathbb{U}}$ commute for every $t > 0$. Thus, we infer from Eqs. (4.5) and (3.29) that

$$\rho(cB) = \bar{\nu}(c \cdot \{t^Q x : x \in \mathbb{S}^{d-1}, t \geq 1\}) = \rho(B)$$

for all $c \in C_{\mathbb{U}}$ and all Borel sets $B \subseteq \mathbb{S}^{d-1}$, i.e., ρ is $C_{\mathbb{U}}$ -invariant. Moreover, [39, Theorem 1.4.12] gives that

$$\begin{aligned}
 \bar{\nu}(C) &= \int_{S_{\mathbb{U}}} \int_0^\infty \mathbb{1}_C(t^Q x) t^{-2} dt \rho(dx) \\
 &= \int_{\mathbb{R}_{>}} \int_{S_{\mathbb{U}}} \mathbb{1}_C(t^Q x) \|t^Q\|^{-\alpha} \rho(dx) \frac{dt}{t} \\
 &= \int_{A_{\mathbb{U}}} \int_{S_{\mathbb{U}}} \mathbb{1}_C(ax) \|a\|^{-\alpha} \rho(dx) H_{A_{\mathbb{U}}}(da).
 \end{aligned} \quad (4.6)$$

Here we used the particular scaling of Q and the fact that dt/t is the push-forward measure of $H_{A_{\mathbb{U}}}$ under $a \mapsto \|a\|$. Thus the implication (i) \Rightarrow (ii) is proved.

For the converse implication, we use that each $u \in \mathbb{U}$ is of the form $u = t^Q c$ for some $t > 0$ and $c \in C_{\mathbb{U}}$. [39, Theorem 1.4.11] gives that if $\bar{\nu}$ satisfies (4.4), then $\bar{\nu}$ is a Lévy measure satisfying $\bar{\nu}((t^Q)^{-1} C) = t \bar{\nu}(C)$. Validity of (3.29) for all $u \in \mathbb{U}$ then follows as in the discrete case. \square

4.4 Matrices invariant under orthogonal transformations

In this section, we analyze the structure of a positive semi-definite $d \times d$ matrix Σ satisfying

$$o \Sigma o^T = \Sigma \quad \text{for all } o \in \mathbb{O}. \quad (4.7)$$

We defined \mathbb{O} as the closed subgroup generated by the $(O(j))_{j \geq 1}$, but it can be any closed subgroup of $\mathbb{O}(d)$.

The main result of this section is the following proposition. To formulate it, we recall two notions. We say that a subspace V of \mathbb{R}^d is \mathbb{O} -invariant if $oV = V$ for all $o \in \mathbb{O}$, and \mathbb{O} -indecomposable if it does not contain any nontrivial \mathbb{O} -invariant subspace.

Proposition 4.4 *Let Σ be a positive semi-definite symmetric matrix, satisfying (4.7). Then there is a decomposition*

$$\mathbb{R}^d = V_+ \oplus V_- \oplus V_1 \oplus \dots \oplus V_l \quad (4.8)$$

into \mathbb{O} -invariant orthogonal subspaces with the following properties:

- (i) *Every $o \in \mathbb{O}$ is the identity mapping on V_+ and minus the identity on V_- . V_+ and V_- are the maximal \mathbb{O} -invariant subspaces with these properties. Further, V_+ and V_- are Σ -invariant subspaces and the restrictions $\Sigma|_{V_{\pm}}$ are positive semi-definite symmetric matrices.*
- (ii) *For each $i = 1, \dots, l$, V_i is \mathbb{O} -indecomposable and Σ -invariant, and $\Sigma|_{V_i}$ is a nonnegative scalar multiple of the identity mapping on V_i .*

One ingredient in the proof of Proposition 4.4 is the following variant of Schur's lemma, see e.g. [57, Corollary XVIII.6.2].

Lemma 4.5 *Let F be a family of real $d \times d$ matrices, and suppose that $\{0\}$ and \mathbb{R}^d are the only subspaces of \mathbb{R}^d that are invariant for each matrix in F . If Σ is a symmetric matrix that commutes with every matrix in F , then $\Sigma = cI_d$ for a constant $c \in \mathbb{R}$.*

For the proof of Proposition 4.4, we need another lemma.

Lemma 4.6 *Let Σ be a positive semi-definite symmetric $d \times d$ matrix and $o \in \mathbb{O}(d)$ such that $\Sigma o = o \Sigma$. Let*

$$\mathbb{R}^d = V_+ \oplus V_- \oplus V_1 \oplus \dots \oplus V_k \quad (4.9)$$

be a decomposition of \mathbb{R}^d into orthogonal o -invariant subspaces where $V_{\pm} = E_{\pm 1}(o)$ and, for each $i = 1, \dots, k$, V_i is a 2-dimensional, o -indecomposable subspace on which o acts as a rotation by an angle $\pi \neq \theta_i \in (0, 2\pi)$.

Then $V_+, V_-, V_1, \dots, V_k$ are Σ -invariant as well.

Proof We have $o^{-1} = o^T$ since $o \in \mathbb{O}(d)$ and multiplication of $\Sigma o = o \Sigma$ with o^T from the left and right yields that o^T and Σ commute. Hence so does $a := (o + o^T)$. As symmetric matrices, a and Σ are diagonalizable. This implies that the eigenspaces of a are Σ -invariant and vice versa. To see this, pick an eigenvector v of a corresponding to the eigenvalue λ and notice that

$$\lambda(\Sigma v) = \Sigma a v = a(\Sigma v),$$

i.e., Σ maps the eigenspace $E_{\lambda}(a)$ into itself.

The eigenvalues of a can easily be computed: Let $b \in \mathbb{O}(d)$ be such that bob^\top is in normal form, i.e.,

$$bob^\top = \begin{pmatrix} I_{d'} & & & 0 \\ & -I_{d''} & & \\ & & R_{\theta_1} & \\ & & & \ddots \\ 0 & & & & R_{\theta_k} \end{pmatrix}$$

for 2×2 -rotation matrices R_{θ_i} , $i = 1, \dots, k$. Then $bo^\top b^\top = (bob^\top)^\top$ is in normal form, too, and $b(o + o^\top)b^\top$ is a diagonal matrix, with diagonal entries $+2$, -2 and $2 \cos(\theta_i)$, $i = 1, \dots, k$. The corresponding eigenspaces are V_+ , V_- and V_i , $i = 1, \dots, k$ if all θ_i are distinct. In this case, the proof is complete.

Suppose there is an $i \in \{1, \dots, k\}$ such that $\theta_i = \theta_j$ for some $j \neq i$. Then $V_i \oplus V_j \subseteq E_{2 \cos(\theta_i)}(a)$. Further,

$$o\Sigma V_i = \Sigma oV_i = \Sigma V_i,$$

i.e., ΣV_i is o -invariant. By the reasoning in the beginning of the proof, $\Sigma V_i \subseteq E_{2 \cos(\theta_i)}(a)$. Hence $\Sigma V_i = \{0\} \subseteq V_i$ or $V_i = V_j$ for some j with $\theta_i = \theta_j$.

We show that $\Sigma V_i = V_j$ for $i \neq j$ is impossible. Choosing a basis of $E_{2 \cos(\theta_i)}(a)$, Σ has to be a symmetric matrix w.r.t. this basis. In particular, it has to permute V_i and V_j . Hence $V_i \oplus V_j$ is Σ -invariant. With respect to a joint basis of V_i and V_j ,

$$\Sigma_{|V_i \oplus V_j} \sim \begin{pmatrix} 0 & A^\top \\ A & 0 \end{pmatrix}$$

for a 2×2 -matrix A with non-vanishing determinant. Thus, $\det(\Sigma_{|V_i \oplus V_j})$ is negative, which violates the Hurwitz criterion for positive semi-definiteness. \square

Proof (Proof of Proposition 4.4) For each $o \in \mathbb{O}$ there is an individual decomposition of the form (4.9), we denote its components by $V_\pm(o)$ etc.

Defining $V_+ := \bigcap_{o \in \mathbb{O}} V_+(o)$ and $V_- := \bigcap_{o \in \mathbb{O}} V_-(o)$, we obtain Σ - and \mathbb{O} -invariant orthogonal subspaces, on which each o acts as the identity or minus the identity, respectively. $\overline{V} := (V_+ \oplus V_-)^\perp$ is Σ - and \mathbb{O} -invariant as well.

We consider the set

$$\mathcal{V} := \{V_1 \oplus \dots \oplus V_l : \{0\} \neq V_j \subseteq \overline{V} \text{ is a } \Sigma\text{- and } \mathbb{O}\text{-invariant subspace}\}.$$

We do not distinguish between decompositions that consist of the same subspaces, but in a different order. \mathcal{V} is non-empty since it contains \overline{V} . The set \mathcal{V} possesses a partial order \prec : a decomposition is larger than another one if the former is a refinement of the latter. Any totally ordered subset of \mathcal{V} has at most d elements since every strict refinement decreases the dimension of at least one subspace by at least 1. Now pick a totally ordered subset of \mathcal{V} with maximal number of elements and within this subset pick the largest element,

$V_1 \oplus \cdots \oplus V_l$, say. Clearly, $V_1 \oplus \cdots \oplus V_l$ is maximal with respect to \prec . We claim that V_1, \dots, V_l are \mathbb{O} -indecomposable. If not, then there is a V_j which contains two proper orthogonal subspaces that are invariant under every $o \in \mathbb{O}$, and hence also Σ -invariant by Lemma 4.6. This contradicts the maximality of the decomposition $V_1 \oplus \cdots \oplus V_l$ with respect to \prec .

Consequently, we can decompose Σ according to a maximal element of \mathcal{V} . In particular, it suffices to solve

$$o|_V \Sigma|_V o|_V^\top = \Sigma|_V \quad \text{for all } o \in \mathbb{O}$$

separately for $V \in \{V_+, V_-, V_1, \dots, V_l\}$.

The restriction of Σ to V_+ or V_- can be any positive semi-definite symmetric matrix, for conjugation by $o|_{V_\pm}$ is the identity mapping for all $o \in \mathbb{O}$. To identify $\Sigma|_{V_i}$, where V_i is an \mathbb{O} -invariant and indecomposable subspace, we use Schur's lemma (Lemma 4.5). This yields that each $\Sigma|_{V_i}$ is a scale multiple of the identity on V_i , and since $\Sigma|_{V_i}$ is positive semi-definite, the scaling factor is nonnegative. \square

A The Choquet-Deny lemma

Given a probability measure μ on the similarity group $\mathbb{S}(d)$, let U be the closed subgroup generated by the support of μ —if μ is the step distribution of the associated multiplicative random walk $(L_n)_{n \in \mathbb{N}_0}$, see Section 3.4, then $U = \mathbb{U}$.

Lemma A.1 *Let $\psi : U \rightarrow \mathbb{R}$ be measurable and bounded. If*

$$\int_U \psi(ug) \mu(du) = \psi(g) \tag{A.1}$$

for all $u \in U$, then ψ is constant μ -a.e.

This is a consequence of [38, Theorem 3]. For the reader's convenience, we state that theorem and show how the lemma can be derived from it.

In the following, let G be a locally compact, separable and unimodular group. A probability measure μ on G is called *aperiodic*, if the closed subgroup generated by the support of μ equals G . Write $[G, G]$ for the commutator subgroup, i.e., the group generated by the commutators $[a, b] := (ba)^{-1}ab$, $a, b \in G$, and $\overline{[G, G]}$ for its closure. Let $H \subsetneq G$ be a normal subgroup of G . Then G acts on H by conjugation (inner automorphisms), i.e.,

$$g.h := g^{-1}hg, \quad g \in G, h \in H.$$

For $A \subseteq H$ write

$$A^G := \{g.a : g \in G, a \in A\}.$$

The action of G on H is said to be *compact* if for each compact $A \subseteq H \setminus \{1_G\}$, 1_G the unit element of G , A^G is relatively compact, i.e., has compact closure. Then [38, Theorem 3] reads as follows:

Theorem A.2 *Let μ be an aperiodic probability measure on G . If $\overline{[G, G]}$ is Abelian or compact and if the action of G on $\overline{[G, G]}$ is compact, then the only bounded, measurable functions ψ satisfying*

$$\int \psi(ug) \mu(du) = \psi(g) \quad \text{for all } g \in G$$

are the μ -almost everywhere constant functions.

Following the proof of [24, Theorem A.1], we show how Theorem A.2 applies to the situation here, i.e., $G = U$ is the closed subgroup generated by the support of μ . Then μ is aperiodic on U by the very definition of U . Referring to Proposition 4.1, there is a closed subgroup A_U of U , which is isomorphic to a closed subgroup of the multiplicative group $\mathbb{R}_{>}$, and a normal compact subgroup $C_U = U \cap \mathbb{O}(d)$, such that $U/C_U \simeq A_U$. The groups A_U and C_U (as a compact group, see [33, Theorem 1.4.1]) are unimodular and hence, by the Fubini formula for the Haar measure on $U = A_U C_U$, [33, Proposition 1.5.5], U is unimodular as well.

Clearly, the commutator subgroup of U is a subgroup of $\mathbb{O}(d)$, hence its closure is compact. Moreover, for any compact $A \subseteq \overline{[U, U]} \setminus \{I_d\}$, A^U is again a subset of $\mathbb{O}(d)$ since

$$u.[a, b] = u^{-1}[a, b]u = o^{-1}[a, b]o,$$

where $u = \|u\|o$ with $o \in C_U \subseteq \mathbb{O}(d)$. Hence, A^U is relatively compact as a subset of a compact set.

B Evaluating the Lévy integrals

In this section, we compute

$$I(x) = \int_{\mathbb{R}^d \setminus \{0\}} \left(e^{i\langle x, y \rangle} - 1 - i\langle x, y \rangle \mathbb{1}_{[0,1]}(|y|) \right) \bar{\nu}(dy), \quad x \in \mathbb{R}^d \quad (\text{B.1})$$

for a deterministic (U, α) -invariant Lévy measure $\bar{\nu}$, i.e. satisfying (3.29).

Lemma B.1 *Let $\bar{\nu}$ be a deterministic Lévy measure satisfying (3.29) for some $1 \neq \alpha \in (0, 2)$, and define $I(x)$ via (B.1). Then, for $0 \neq x \in \mathbb{R}^d$,*

$$I(x) = -|x|^\alpha \eta_1^\alpha(x) + i|x|^\alpha \eta_2^\alpha(x) + i\langle x, \gamma^\alpha \rangle, \quad (\text{B.2})$$

with functions $\eta_1^\alpha, \eta_2^\alpha$ defined in (1.19) and (1.20), respectively. η_1^α and η_2^α are bounded real functions satisfying $\eta_j^\alpha(u^\top x) = \eta_j^\alpha(x)$ for all $u \in \mathbb{U}$, $x \in \mathbb{R}^d \setminus \{0\}$, $j = 1, 2$, and η_1^α is nonnegative. The vector γ^α satisfies $\sigma\gamma^\alpha = \gamma^\alpha$ for all $\sigma \in C_U$.

Proof Fix $0 \neq x \in \mathbb{R}^d$ and notice that $I(x)$ is finite since $\bar{\nu}$ is a Lévy measure. Further, according to Proposition 4.3, there is a C_U -invariant finite measure ρ on S_U such that (4.4) holds.

We set

$$\gamma^\alpha := \begin{cases} -\int y \mathbb{1}_{[0,1]}(|y|) \bar{\nu}(dy) = -\int_{A_U} \int_{S_U} ax \mathbb{1}_{[0,1]}(|ax|) \|a\|^{-\alpha} \rho(dx) H_{A_U}(da) & \text{if } \alpha < 1 \\ -\int_{A_U} \int_{S_U} ax \mathbb{1}_{(1,\infty)}(|ax|) \|a\|^{-\alpha} \rho(dx) H_{A_U}(da), & \text{if } \alpha \in (1, 2). \end{cases}$$

The asserted C_U -invariance follows from the C_U -invariance of ρ .

Recalling the definitions

$$\begin{aligned} \eta_1^\alpha(x) &= \frac{1}{|x|^\alpha} \int (1 - \cos(\langle x, y \rangle)) \nu^\alpha(dy) \\ \eta_2^\alpha(x) &= \frac{1}{|x|^\alpha} \int (\sin(\langle x, y \rangle) - \mathbb{1}_{\{\alpha > 1\}} \langle x, y \rangle) \nu^\alpha(dy), \end{aligned}$$

(B.2) holds, and it remains to prove boundedness and invariance properties of η_i^α , $i = 1, 2$. Let $u \in \mathbb{U}$, then, using (3.29),

$$\begin{aligned} \eta_1^\alpha(u^\top x) &= \frac{1}{\|u\|^\alpha |x|^\alpha} \int (1 - \cos(\langle x, uy \rangle)) \nu^\alpha(dy) \\ &= \frac{\|u\|^\alpha}{\|u\|^\alpha |x|^\alpha} \int (1 - \cos(\langle x, y \rangle)) \nu^\alpha(dy) = \eta_1^\alpha(x), \end{aligned}$$

and the invariance of η_2^α is proved along the same lines. This implies in particular that the continuous functions η_i^α are determined by their respective values on the (relative) compact set S_U , hence the asserted boundedness follows. \square

For $\alpha = 1$, we can compute a meaningful expression for $\eta^1(x)$ only for $x \in E_1(Q^\top)$. Notice that this implies $x \in E_1(t^Q)$ for all $t \geq 0$. Hence, using formula (4.6) for $\bar{\nu}$, we obtain

$$\begin{aligned} \eta^1(x) &= \int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}_>} \left(e^{i\langle (t^Q)^\top x, s \rangle} - 1 - i\langle (t^Q)^\top x, s \rangle \mathbb{1}_{\{|(t^Q)^\top s| \leq 1\}} \right) t^{-2} dt \rho(ds) \\ &= \int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}_>} \left(e^{i\langle tx, s \rangle} - 1 - i\langle tx, s \rangle \mathbb{1}_{\{|s| \leq 1\}} \right) t^{-2} dt \rho(ds) \\ &= - \int_{\mathbb{S}^{d-1}} \left(|\langle x, s \rangle| + i \frac{2}{\pi} \langle x, s \rangle \log |\langle x, s \rangle| \right) \rho(ds) + i\langle \gamma, x \rangle \end{aligned} \quad (\text{B.3})$$

for a suitable $\gamma \in \mathbb{R}^d$, see [72, Theorem 14.10] for details.

C Auxiliary results from (Markov) renewal theory

Lemma C.1 *Let $(S_n)_{n \in \mathbb{N}_0}$ be a random walk with i.i.d. increments, $S_0 = 0$, $\mathbb{E}[S_1] > 0$ and $\mathbb{E}[(S_1^+)^2] < \infty$. Let $\tau(t) := \inf\{n \in \mathbb{N}_0 : S_n > t\}$, $t \geq 0$. Then, for every $0 < a < 1$,*

$$t\mathbb{P}(S_{\tau(t)-1} < at) \rightarrow 0 \quad \text{as } t \rightarrow \infty. \quad (\text{C.1})$$

Proof $\mathbb{E}[(S_1^+)^2] < \infty$ implies $\lim_{t \rightarrow \infty} t^2 \mathbb{P}(S_1 > t) = 0$. Further, it is known from standard random walk theory that $\lim_{t \rightarrow \infty} t^{-1} \mathbb{E}[\tau(t)] = \mathbb{E}[S_1]^{-1}$. Consequently, setting, for $n \in \mathbb{N}$, $A_n := \{S_0 \leq t, \dots, S_{n-2} \leq t, S_{n-1} < at\}$, we have

$$\begin{aligned} t\mathbb{P}(S_{\tau(t)-1} < at) &= t \sum_{n \geq 1} \mathbb{P}(A_n \cap \{S_{n+1} > t\}) \leq t \sum_{n \geq 1} \mathbb{P}(A_n) \mathbb{P}(S_1 > (1-a)t) \\ &\leq t^2 \mathbb{P}(S_1 > (1-a)t) \cdot \frac{1}{t} \sum_{n \geq 1} \mathbb{P}(\tau(t) \geq n) \\ &= t^2 \mathbb{P}(S_1 > (1-a)t) \cdot \frac{\mathbb{E}[\tau(t)]}{t} \rightarrow 0 \quad \text{as } t \rightarrow \infty. \end{aligned}$$

□

A rate-of-convergence result in Markov renewal theory

Throughout this section, let $((O_n, S_n))_{n \in \mathbb{N}_0}$ be a random walk on $\mathbb{O}(d) \times \mathbb{R}$ with increment law μ , say. The components O_n and S_n may be dependent. We assume that μ satisfies the minorization condition (M) and that

$$\mathbb{E}[S_1] > 0 \text{ and } \mathbb{E}[|S_1|^{\ell+1+\delta}] < \infty \quad (\text{C.2})$$

for some $\ell > 0$. Let \mathbb{O} be the closed subgroup of $\mathbb{O}(d)$ generated by the support of O_1 .

Proposition C.2 *Let $\ell > 0$ and $g : \mathbb{R} \rightarrow [0, \infty)$ be a measurable function that is decreasing on $[0, \infty)$, with $g(t) = 0$ for all $t < 0$ and $\lim_{t \rightarrow \infty} t^{\ell+1+\epsilon} g(t) = 0$ for some $0 < \epsilon < (\delta \wedge 1)$. Then*

$$\lim_{t \rightarrow \infty} \sup_{|f| \leq g} t^{\ell+\epsilon} \left| \mathbb{E} \left[\sum_{n=0}^{\infty} f(O_n, t - S_n) \right] - \frac{1}{\mathbb{E}[S_1]} \int_0^\infty \int_{\mathbb{O}} f(o, r) H_{\mathbb{O}}(do) dr \right| = 0.$$

Here and below, $\sup_{|f| \leq g}$ means the supremum over all measurable functions $f : \mathbb{O} \times \mathbb{R} \rightarrow \mathbb{R}$ satisfying $\sup_{o \in \mathbb{O}} |f(o, x)| \leq g(x)$ for all $x \in \mathbb{R}$.

The main ingredient in the proof will be the use of regeneration techniques for general state space Markov chains as developed in [9, 67]. We sum up what is needed in the subsequent lemma.

Lemma C.3 *There is a measurable space (Ω, \mathcal{G}) together with a family of probability measures $(\mathbb{P}_{o,r})_{o \in \mathbb{O}, r \in \mathbb{R}}$ and sequences of random variables $((M_n, R_n))_{n \geq 0}$ and $(\tau_n)_{n \geq 1}$ with*

$$\mathbb{P}_{o,r}(((M_n, R_n))_{n \geq 0} \in \cdot) = \mathbb{P}(((oO_n, r + S_n))_{n \geq 0} \in \cdot) \quad (\text{C.3})$$

for all $o \in \mathbb{O}$, $r \in \mathbb{R}$. Further, the following properties hold:

- (i) *There is a filtration $(\mathcal{G}_n)_{n \in \mathbb{N}}$ such that $((M_n, R_n))_{n \geq 0}$ is Markov adapted to $(\mathcal{G}_n)_{n \in \mathbb{N}}$, and $(\tau_n)_{n \geq 1}$ is a sequence of predictable $(\mathcal{G}_n)_{n \in \mathbb{N}}$ -stopping times, i.e., $\{\tau_n = k\} \in \mathcal{G}_{k-1}$.*
- (ii) *There are probability measures ν on \mathbb{O} and η on \mathbb{R} , η having a bounded Lebesgue density, such that for all $n \geq 1$ and $o \in \mathbb{O}$, under \mathbb{P}_o , $((M_{\tau_n+k}, R_{\tau_n+k} - R_{\tau_n-1}))_{0 \leq k \leq \tau_{n+1} - \tau_n - 1}$ is independent of $(M_0, R_0, \dots, M_{\tau_n-1}, R_{\tau_n-1})$ and has law $\mathbb{P}_{\nu \otimes \eta}(((M_k, R_k))_{k=0}^{\tau_{n+1}-1} \in \cdot)$.*
- (iii) *For each bounded measurable function $f : \mathbb{O} \rightarrow \mathbb{R}$,*

$$\mathbb{E}_{\nu \otimes \eta} \left[\sum_{n=0}^{\tau_1-1} f(M_n) \right] = \mathbb{E}_{\nu \otimes \eta}[\tau_1] \int_{\mathbb{O}} f(o) H_{\mathbb{O}}(do). \quad (\text{C.4})$$

- (iv) *There are $C, \lambda > 0$ such that $\mathbb{P}_{o,r}(\tau_1 > n) \leq Ce^{-\lambda n}$ for all $o \in \mathbb{O}$, $r \in \mathbb{R}$ and $n \in \mathbb{N}$.*

Here and below, we use the shorthand $\mathbb{P}_{\nu \otimes \eta} = \int_{\mathbb{O}} \int_{\mathbb{R}} \mathbb{P}_{o,r} \nu(do) \eta(dr)$, the same notation for expectations, and sometimes omit the initial value R_0 if it is irrelevant.

Now we turn to the proof of Proposition C.2.

Proof (Proof of Proposition C.2) In order to prove this result, we will combine methods from [8] and [68]. Below, we describe the steps of the proof and defer the technicalities to several lemmata. By splitting f into its positive and negative part, it suffices to consider nonnegative functions which are bounded by g .

Let $((M_n, R_n))_{n \geq 0}$ and $(\tau_n)_{n \geq 0}$ be as in Lemma C.3. Define $V_0 := 0$ and

$$V_n := \sum_{k=1}^n (R_{\tau_{k+1}-1} - R_{\tau_k-1}) = R_{\tau_{n+1}-1} - R_{\tau_1-1}.$$

Then, under each $\mathbb{P}_{o,r}$, $(V_n)_{n \geq 1}$ is a random walk with i.i.d. increments and increment law $\mathbb{P}_{\nu \otimes \eta}(R_{\tau_1-1} \in \cdot)$ which is absolutely continuous since the law η of R_0 is absolutely continuous. Since $\mathbb{E}[S_1] > 0$ and

$$1 = \mathbb{P}(S_n/n \rightarrow \mathbb{E}[S_1] \text{ as } n \rightarrow \infty) = \mathbb{P}_{I_d,0}(R_n/n \rightarrow \mathbb{E}[S_1] \text{ as } n \rightarrow \infty),$$

we deduce that $\mathbb{E}_{\nu \otimes \eta}[V_1] = \mathbb{E}_{\nu \otimes \eta}[\tau_1] \mathbb{E}[S_1]$.

Let f be nonnegative and set

$$\hat{f}(t) := \mathbb{E}_{\nu \otimes \eta} \left[\sum_{k=0}^{\tau_1-1} f(M_k, t - R_k) \right].$$

Then we can proceed as in [8, Section 4] to obtain

$$\begin{aligned} \mathbb{E} \left[\sum_{n=0}^{\infty} f(O_n, t - S_n) \right] &= \mathbb{E}_{I_d,0} \left[\sum_{n=0}^{\infty} f(M_n, t - R_n) \right] \\ &= \mathbb{E}_{I_d,0} \left[\sum_{n=0}^{\tau_1-1} f(M_n, t - R_n) \right] \\ &\quad + \mathbb{E}_{I_d,0} \left[\sum_{n=1}^{\infty} \mathbb{E} \left[\sum_{k=\tau_n}^{\tau_{n+1}-1} f(M_k, t - (R_k - R_{\tau_n-1}) - R_{\tau_n-1}) \middle| \mathcal{G}_{\tau_n-1} \right] \right] \\ &= \mathbb{E}_{I_d,0} \left[\sum_{n=0}^{\tau_1-1} f(M_n, t - R_n) \right] + \mathbb{E}_{I_d,0} \left[\sum_{n=0}^{\infty} \hat{f}(t - R_{\tau_{n+1}-1} - V_n) \right] =: E_1(t) + E_2(t). \end{aligned} \quad (\text{C.5})$$

We show in Lemma C.4 that $t^{\ell+\epsilon}E_1(t)$ tends to zero, uniformly over all f with $|f| \leq g$. We rewrite

$$E_2(t) = \int_{\mathbb{R}} \mathbb{E}_{\nu \otimes \eta} \left[\sum_{n=0}^{\infty} \hat{f}(t-s-V_n) \right] \mathbb{P}_{I_d,0}(R_{\tau_1-1} \in ds)$$

and use (C.4) to infer that (recall that $f \geq 0$)

$$\begin{aligned} \frac{1}{\mathbb{E}_{\nu \otimes \eta}[V_1]} \int_{\mathbb{R}} \hat{f}(r) dr &= \frac{1}{\mathbb{E}_{\nu \otimes \eta}[V_1]} \mathbb{E}_{\nu \otimes \eta} \left[\sum_{k=0}^{\tau_1-1} \int_{\mathbb{R}} f(M_k, r - R_k) dr \right] \\ &= \frac{1}{\mathbb{E}_{\nu \otimes \eta}[V_1]} \mathbb{E}_{\nu \otimes \eta} \left[\sum_{k=0}^{\tau_1-1} \int_{\mathbb{R}} f(M_k, r) dr \right] \\ &= \frac{1}{\mathbb{E}[S_1]} \int_0^{\infty} \int_{\mathbb{O}} f(o, r) H_{\mathbb{O}}(do) dr. \end{aligned}$$

Then the claimed convergence rate holds, if

$$\int_{\mathbb{R}} \sup_{|f| \leq g} t^{\ell+\epsilon} \left| \mathbb{E}_{\nu \otimes \eta} \left[\sum_{n=0}^{\infty} \hat{f}(t-s-V_n) \right] - \frac{1}{\mathbb{E}_{\nu \otimes \eta}[V_1]} \int_0^{\infty} \hat{f}(r) dr \right| \mathbb{P}_{I_d,0}(R_{\tau_1-1} \in ds)$$

tends to 0 as $t \rightarrow \infty$. This result will be established in Lemma C.5. \square

Lemmata needed in the proof of Proposition C.2

Proof (Proof of Lemma C.3) Observe that $((oO_n, r + S_n))_{n \in \mathbb{N}_0}$ is indeed a Markov chain on $\mathbb{O} \times \mathbb{R}$ and that the increments of $S_n - S_{n-1}$ are independent of the past. Since μ satisfies the minorization condition (M), we have that for all $o \in \text{SO}(d)$, $B \in \text{SO}(d)$,

$$\mathbb{P}(oO_1 \in B) = \mathbb{P}(O_1 \in o^{-1}B) \geq \gamma H_{\mathbb{O}}((o^{-1}B) \cap \text{SO}(d)) \geq \frac{\gamma}{2} H_{\text{SO}(d)}(B). \quad (\text{C.6})$$

If $\mathbb{O} = \text{SO}(d)$, this shows that $(oO_n)_{n \in \mathbb{N}}$ is a Doeblin chain on \mathbb{O} (see [65, Section 16.2] for the definition).

If \mathbb{O} contains elements with determinant -1 as well, then readily $\mathbb{O} = \mathbb{O}(d)$, for $\text{SO}(d) \subseteq \mathbb{O}$, and the product of two matrices with negative determinant has a positive determinant. Moreover, this necessitates $\mathbb{P}(\det(O_1) = -1) > 0$. Then, for all $o \in \mathbb{O} \setminus \text{SO}(d)$ and $B \subseteq \text{SO}(d)$,

$$\begin{aligned} \mathbb{P}(oO_2 \in B) &\geq \int \mathbb{1}_{\{\det(o') = -1\}} \mathbb{P}(O_1 \in (oo')^{-1}B) \mathbb{P}(O_1 \in do') \\ &\geq \mathbb{P}(\det(O_1) = -1) \frac{\gamma}{2} H_{\text{SO}(d)}(B). \end{aligned} \quad (\text{C.7})$$

Thus, $(oO_n)_{n \in \mathbb{N}}$ is a Doeblin chain in the case $\mathbb{O} = \mathbb{O}(d)$, too. Its unique invariant probability measure is given by the normalized Haar measure $H_{\mathbb{O}}$ on $\mathbb{O}(d)$.

Again by the minorization condition for μ , it follows that there is an absolutely continuous measure $\eta(dx) := \mathbb{1}_I(x)dx$ such that

$$\mathbb{P}(oO_1 \in A, S_1 \in B) \geq \frac{\gamma}{2} H_{\text{SO}(d)}(A) \eta(B) \quad (\text{C.8})$$

for all $o \in \text{SO}(d)$ and all measurable A, B .

(C.6) and (C.7) yield that there is some $q < 1$ such that, for all $o \in \mathbb{O}$,

$$\mathbb{P}(oO_k \notin \text{SO}(d) \text{ for } k = 1, 2) \leq q. \quad (\text{C.9})$$

It follows that $(oO_n)_{n \in \mathbb{N}}$ is $(\text{SO}(d), \gamma/2, \nu, 1)$ -recurrent in the sense of [8], with $\nu := H_{\text{SO}(d)}$. Then, $((M_n, R_n))_{n \in \mathbb{N}_0}$ can be constructed along the same lines as in [8, Section 3]: Under $\mathbb{P}_{o,r}$, let $((M_n, R_n))_{n \in \mathbb{N}}$ have the same transitions as $(oO_n, r + S_n)_{n \in \mathbb{N}}$, but whenever O_n

enters $\text{SO}(d)$, an independent $B(1, \gamma/2)$ -distributed coin is flipped. If 1 shows up, then $(M_{n+1}, R_{n+1} - R_n)$ is generated according to $\nu \otimes \eta$, this event we call a *regeneration*; if 0 shows up, then $(M_{n+1}, R_{n+1} - R_n)$ is generated according to $(1 - \frac{\gamma}{2})^{-1} (\mathbb{P}((M_n O_1, S_1) \in \cdot) - \frac{\gamma}{2} \nu \otimes \eta)$. Thus, the total transition probabilities are still equal to that of $(oO_n, r + S_n)_{n \in \mathbb{N}}$. Let $\tau_0 = 0$ and τ_n be the n th regeneration time, see [9, 67] for details. This gives Assertions 1 and 2, while Assertion 3 is proved in [9, Theorem 6.1]. The construction together with (C.9) show that at least every third step, there is a uniform positive chance for regeneration, this yields Assertion 4. \square

Lemma C.4 *Let g be as in Proposition C.2 and $((M_n, R_n))_{n \geq 0}$ as in Lemma C.3. Then*

$$\lim_{t \rightarrow \infty} \sup_{|f| \leq g} t^{\ell+\epsilon+1} \left| \mathbb{E}_{I_d,0} \left[\sum_{n=0}^{\tau_1-1} f(M_n, t - R_n) \right] \right| = 0. \quad (\text{C.10})$$

In particular, $\lim_{t \rightarrow \infty} t^{\ell+\epsilon+1} \hat{g}(t) = 0$. Moreover,

$$\lim_{t \rightarrow \infty} t^{\ell+\epsilon+1} \mathbb{P}_{I_d,0}(R_{\tau_1-1} > t/2) = 0. \quad (\text{C.11})$$

Proof In order to prove (C.10), we assume that $g(0) \leq 1$ and fix some $|f| \leq g$. Recall that, by Lemma C.3(iv), $\mathbb{P}_{I_d,0}(\tau_1 > n) \leq C e^{-\lambda n}$ for some $C, \lambda > 0$. Define $n_t = (\log t)(\ell + 2)/\lambda$, $t > 0$. Then

$$\begin{aligned} t^{\ell+\epsilon+1} \left| \mathbb{E}_{I_d,0} \left[\sum_{n=0}^{\tau_1-1} f(M_n, t - R_n) \right] \right| &\leq t^{\ell+\epsilon+1} \mathbb{E}_{I_d,0} \left[\sum_{n=0}^{\tau_1-1} g(t - R_n) \right] \\ &\leq t^{\ell+\epsilon+1} \mathbb{E}_{I_d,0} \left[\mathbb{1}_{\{\tau_1 \leq n_t\}} \sum_{n=0}^{\lfloor n_t \rfloor} g(t - R_n) (\mathbb{1}_{\{R_n \leq t/2\}} + \mathbb{1}_{\{R_n > t/2\}}) \right] \\ &\quad + t^{\ell+\epsilon+1} \mathbb{E}_{I_d,0} \left[\mathbb{1}_{\{\tau_1 > n_t\}} \sum_{n=0}^{\tau_1-1} g(t - R_n) \right] =: I_1(t) + I_2(t) + I_3(t). \end{aligned}$$

Recall that g is decreasing on $[0, \infty)$ and $\lim_{t \rightarrow \infty} t^{\ell+1+\delta} g(t) = 0$ for some $\delta > 0$. This gives

$$I_1(t) = t^{\ell+\epsilon+1} \mathbb{E}_{I_d,0} \left[\mathbb{1}_{\{\tau_1 \leq n_t\}} \sum_{n=0}^{\lfloor n_t \rfloor} g(t - R_n) \mathbb{1}_{\{R_n \leq t/2\}} \right] \leq t^{\ell+\epsilon+1} (\lfloor n_t \rfloor + 1) g(t/2) \xrightarrow{t \rightarrow \infty} 0.$$

By Jensen's inequality, $\mathbb{E}[|S_n|^\kappa] \leq n^{\kappa-1} \mathbb{E}[|S_1|^\kappa]$ for all $\kappa \geq 1$. Thus, applying Markov's inequality,

$$\begin{aligned} I_2(t) &\leq t^{\ell+\epsilon+1} \sum_{n=0}^{\lfloor n_t \rfloor} \mathbb{P}(S_n > t/2) \leq t^{\ell+\epsilon+1} \sum_{n=0}^{\lfloor n_t \rfloor} \frac{\mathbb{E}[|S_n|^{\ell+1+\delta}]}{(t/2)^{\ell+1+\delta}} \\ &\leq \frac{2^{\ell+1+\delta}}{t^{\delta-\epsilon}} \sum_{n=0}^{\lfloor n_t \rfloor} n^{\ell+\delta} \mathbb{E}[|S_1|^{\ell+1+\delta}] \leq 2^{\ell+1+\delta} \frac{n_t^{\ell+1+\delta}}{t^{\delta-\epsilon}} \mathbb{E}[|S_1|^{\ell+1+\delta}], \end{aligned}$$

which tends to zero as $t \rightarrow \infty$. Finally,

$$\begin{aligned} I_3(t) &\leq t^{\ell+\epsilon+1} \mathbb{E}_{I_d,0} [\tau_1 \mathbb{1}_{\{\tau_1 > n_t\}}] \\ &\leq t^{\ell+\epsilon+1} \left(n_t \mathbb{P}_{I_d,0}(\tau_1 > n_t) + \int_{n_t}^{\infty} \mathbb{P}_{I_d,0}(\tau_1 > r) dr \right) \\ &\leq t^{\ell+\epsilon+1} \left(n_t C e^{-\lambda n_t} + \frac{C}{\lambda} e^{-\lambda n_t} \right) \\ &\leq C \frac{t^{\ell+\epsilon+1} (n_t + 1/\lambda)}{t^{\ell+2}} \rightarrow 0 \end{aligned}$$

as $t \rightarrow \infty$. Thus we have proved the first and second assertion. (C.11) follows from (C.10) with $g(s) = \mathbb{1}_{[0, t/2)}(s)$. \square

Lemma C.5 *It holds that*

$$\lim_{t \rightarrow \infty} \int_{\mathbb{R}} \sup_{|f| \leq g} t^{\ell+\epsilon} \left| \mathbb{E}_{\nu \otimes \eta} \left[\sum_{n=0}^{\infty} \hat{f}(t-s-V_n) \right] - \frac{\int_0^{\infty} \hat{f}(r) dr}{\mathbb{E}_{\nu \otimes \eta}[V_1]} \right| \mathbb{P}_{I_d,0}(R_{\tau_1-1} \in ds) = 0. \quad (\text{C.12})$$

Proof We start by proving that the functions \hat{f} are uniformly directly Riemann-integrable over $|f| \leq g$. Write

$$\hat{f}(t) = \int \mathbb{E}_{\nu \otimes \delta_0} \left[\sum_{k=0}^{\tau_1-1} f(M_k, t-s-R_k) \right] \eta(ds),$$

then observe that

$$\int \left| \mathbb{E}_{\nu \otimes \delta_0} \left[\sum_{k=0}^{\tau_1-1} f(M_k, t-R_k) \right] \right| dt \leq \mathbb{E}_{\nu \otimes \delta_0} \left[\sum_{k=0}^{\tau_1-1} \int g(t-R_k) dt \right] = \mathbb{E}_{\nu \otimes \delta_0}[\tau_1] \int g(t) dt$$

and recall that η has a bounded Lebesgue density. Consequently, \hat{f} is the convolution of a Lebesgue integrable function with a bounded function and hence continuous, see [7, Lemma VII.1.2]. Further, arguing as in [8, Eq. (4.4)],

$$\begin{aligned} \sum_{l \in \mathbb{Z}} \sup_{t \in [lh, (l+1)h]} |\hat{f}(t)| &\leq \mathbb{E}_{\nu \otimes \eta}[\tau_1] \int_{\mathbb{O}} \sum_{l \in \mathbb{Z}} \sup_{t \in [l2h, (l+1)2h]} |f(o, t)| H_{\mathbb{O}}(do) \\ &\leq \mathbb{E}_{\nu \otimes \eta}[\tau_1] \sum_{l \in \mathbb{Z}} \sup_{t \in [l2h, (l+1)2h]} g(t) =: C_g. \end{aligned}$$

Hence, it suffices to show that g is directly Riemann-integrable. The latter is clear since g is monotone and Lebesgue-integrable (see [7, Proposition V.4.1(v)]). We have the uniform bound (cf. [7, Theorem V.2.4(iii)])

$$\sup_{t \in \mathbb{R}} \sup_{|f| \leq g} \mathbb{E}_{\nu \otimes \eta} \left[\sum_{n=0}^{\infty} |\hat{f}(t-V_n)| \right] \leq C_g \mathbb{E}_{\nu \otimes \eta} \left[\sum_{n=0}^{\infty} \mathbb{1}_{[-4h, 4h]}(V_n) \right] =: D_g < \infty.$$

Now we decompose the integral inside the limit in (C.12) according to the set $\{R_{\tau_1-1} > t/2\}$ to obtain the following upper bound

$$\begin{aligned} \sup_{s \leq t/2} \sup_{|f| \leq g} t^{\ell+\epsilon} \left| \mathbb{E}_{\nu \otimes \eta} \left[\sum_{n=0}^{\infty} \hat{f}(t-s-V_n) \right] - \frac{1}{\mathbb{E}_{\nu \otimes \eta}[V_1]} \int_0^{\infty} \hat{f}(r) dr \right| \\ + 2D_g t^{\ell+\epsilon} \mathbb{P}_{I_d,0}(R_{\tau_1-1} > t/2). \end{aligned}$$

The second term tends to zero by (C.11). For the first term, we invoke [68, Theorem 4.2(ii)] (with $G = \delta_0$), which gives (note that $|f| \leq g$ implies $|\hat{f}| \leq \hat{g}$)

$$\lim_{t \rightarrow \infty} t^{\ell+\epsilon} \sup_{|\hat{f}| \leq \hat{g}} \left| \mathbb{E}_{\nu \otimes \eta} \left[\sum_{n=0}^{\infty} \hat{f}(t-s-V_n) \right] - \frac{1}{\mathbb{E}_{\nu \otimes \eta}[V_1]} \int_0^{\infty} \hat{f}(r) dr \right| = 0$$

as soon as V_1 has positive drift (here, $\mathbb{E}_{\nu \otimes \eta}[V_1] = \mathbb{E}_{\nu \otimes \eta}[\tau_1] \mathbb{E}[S_1] > 0$ by the proof of Proposition C.2), a spread-out law (here, the law of V_1 is even absolutely continuous) and $\mathbb{E}_{\nu \otimes \eta}[|V_1|^{\ell+\epsilon+1}] < \infty$ (which is true by (C.2) and Lemma C.3(iv)) and \hat{g} is bounded, Lebesgue-integrable and satisfies

$$t^{\ell+\epsilon} \int_t^{2t} \hat{g}(r) dr \rightarrow 0 \quad \text{and} \quad t^{\ell+\epsilon} \sup_{r \geq t} \hat{g}(r) \rightarrow 0 \quad \text{as } t \rightarrow \infty. \quad (\text{C.13})$$

Lemma C.4 gives $\lim_{t \rightarrow \infty} t^{\ell+\epsilon+1} \hat{g}(t) = 0$, which is sufficient for (C.13) to hold. \square

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